1. Niz $(a_n)_{n\in\mathbb{N}}$ zadan je pravilom $a_n = \dfrac{2\cdot n+3}{n+1}$.

a) Izračunajte graničnu vrijednost L zadanoga niza.

b) Odredite najmanji $n \in \mathbb{N}$ za koji je $|a_n - L| < 10^{-5}$.

2. Niz $(b_n)_{n\in\mathbb{N}}$ zadan je pravilom $b_n = \dfrac{1-4\cdot n}{n+2}$.

a) Izračunajte graničnu vrijednost L zadanoga niza.

b) Odredite najmanji $n \in \mathbb{N}$ za koji je $|b_n - L| < 10^{-5}$.

3. Izračunajte graničnu vrijednost niza $(c_n)_{n\in\mathbb{N}}$ čiji je opći član definiran pravilom $c_n = \left(1-\dfrac{2}{n}\right)^{-n}$.

4. Izračunajte graničnu vrijednost niza $(d_n)_{n\in\mathbb{N}}$ čiji je opći član definiran pravilom $d_n = \dfrac{1}{e^2}\cdot\left(\dfrac{2\cdot n+3}{2\cdot n+1}\right)^{2\cdot n}$.

5. Izračunajte graničnu vrijednost niza $(e_n)_{n\in\mathbb{N}}$ čiji je opći član definiran pravilom $e_n = \sqrt{4\cdot n^2+8\cdot n+7} - 2\cdot n$.

6. Izračunajte graničnu vrijednost niza $(f_n)_{n\in\mathbb{N}}$ čiji je opći član definiran pravilom $f_n = 3\cdot n - \sqrt{9\cdot n^2 - 36\cdot n + 37}$.

7. Izračunajte graničnu vrijednost $\lim\limits_{x\to+\infty}\left(2\cdot x - \sqrt{4\cdot x^2 - 8\cdot x + 19}\right)$.

8. Izračunajte graničnu vrijednost $\lim\limits_{x\to-\infty}\left(x + \sqrt{x^2 - x + 1}\right)$.

9. Izračunajte graničnu vrijednost $\lim\limits_{x\to-\infty}\dfrac{(x+1)^2+(x-1)^2}{(1-x)\cdot(x+1)}$.

10. Izračunajte graničnu vrijednost $\lim\limits_{t\to+\infty}\dfrac{(2\cdot t+3)^2+(3\cdot t-2)^2}{(7\cdot t-8)^2-(6\cdot t+5)^2}$.

11. Izračunajte graničnu vrijednost $\lim\limits_{x\to 0}\left[(1+2\cdot x)^{\frac{1}{3\cdot x}}\right]$.

12. Izračunajte graničnu vrijednost $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x^2}\right)^{(2-x)\cdot(x+2)}$.

REZULTATI ZADATAKA

1. **a)** $L = 2$; **b)** $n = 100\,000$.

2. **a)** $L = -4$; **b)** $n = 899\,999$.

3. e^2.

4. 1.

5. 2.

6. 6.

7. 2.

8. $-\frac{1}{2}$.

9. -2.

10. 1.

11. $e^{\frac{2}{3}} = \sqrt[3]{e^2}$.

12. $e^{-1} = \frac{1}{e}$.

UPUTE I RJEŠENJA NEKIH ZADATAKA

1. a) $L=\lim_n a_n=\lim_n\left(\frac{2\cdot n+3}{n+1}\right)=\lim_n\left(\frac{\frac{2\cdot n+3}{n}}{\frac{n+1}{n}}\right)=\lim_n\left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right)=\frac{2+0}{1+0}=2.$

b) $|a_n-L|<10^{-5}\Leftrightarrow\left|\frac{2\cdot n+3}{n+1}-2\right|<10^{-5}\Leftrightarrow\left|\frac{2\cdot n+3-2\cdot(n+1)}{n+1}\right|<10^{-5}\Leftrightarrow$

$\Leftrightarrow\left|\frac{1}{n+1}\right|<10^{-5}\Rightarrow\frac{1}{n+1}<10^{-5}\Leftrightarrow n+1>\frac{1}{10^{-5}}\Leftrightarrow n>99\ 999\Rightarrow n_{\min}=100\ 000.$

2. Rješenje analogno kao u Zadatku 1.

3. $L=\lim_n c_n=\lim_n\left[\left(1-\frac{2}{n}\right)^{-n}\right]=\lim_n\left\{\left[1+\frac{(-2)}{n}\right]^n\right\}^{-1}=\left\{\lim_n\left[1+\frac{(-2)}{n}\right]^n\right\}^{-1}=\left(e^{-2}\right)^{-1}=e^2.$

4. Zamijenimo $t:=2\cdot n+1$. Tada su $2\cdot n+3=t+2$ i $2\cdot n=t-1$. Kad $n\to+\infty$, onda i $t\to+\infty$. Tako dobivamo:

$$L=\frac{1}{e^2}\cdot\lim_t\left(\frac{t+2}{t}\right)^{t-1}=\frac{1}{e^2}\cdot\lim_t\left(1+\frac{2}{t}\right)^{t-1}=\frac{1}{e^2}\cdot\left[\lim_t\left(1+\frac{2}{t}\right)^{t}\right]\cdot\left[\lim_t\left(1+\frac{2}{t}\right)^{-1}\right]=\frac{1}{e^2}\cdot e^2\cdot 1^{-1}=1.$$

5. $L=\lim_n e_n=\lim_n\left(\sqrt{4\cdot n^2+8\cdot n+7}-2\cdot n\right)=\lim_n\left[\left(\sqrt{4\cdot n^2+8\cdot n+7}-2\cdot n\right)\cdot\frac{\left(\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n\right)}{\left(\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n\right)}\right]=$

$$=\lim_n\left[\frac{\left(\sqrt{4\cdot n^2+8\cdot n+7}\right)^2-(2\cdot n)^2}{\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n}\right]=\lim_n\left[\frac{4\cdot n^2+8\cdot n+7-4\cdot n^2}{\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n}\right]=\lim_n\left(\frac{8\cdot n+7}{\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n}\right)=$$

$$=\lim_n\frac{\frac{8\cdot n+7}{n}}{\frac{\sqrt{4\cdot n^2+8\cdot n+7}+2\cdot n}{n}}=\lim_n\frac{8+\frac{7}{n}}{\sqrt{\frac{4\cdot n^2+8\cdot n+7}{n^2}}+2}=\lim_n\frac{8+\frac{7}{n}}{\sqrt{4+\frac{8}{n}+\frac{7}{n^2}}+2}=\frac{8+0}{\sqrt{4+0+0}+2}=2.$$

6. Rješenje analogno kao u Zadatku 5.

7. $L=\lim_{x\to+\infty}\left(2\cdot x-\sqrt{4\cdot x^2-8\cdot x+19}\right)=\lim_{x\to+\infty}\left[\frac{\left(2\cdot x-\sqrt{4\cdot x^2-8\cdot x+19}\right)\cdot\left(2\cdot x+\sqrt{4\cdot x^2-8\cdot x+19}\right)}{2\cdot x+\sqrt{4\cdot x^2-8\cdot x+19}}\right]=$

$$=\lim_{x\to+\infty}\left[\frac{4\cdot x^2-\left(4\cdot x^2-8\cdot x+19\right)}{2\cdot x+\sqrt{4\cdot x^2-8\cdot x+19}}\right]=\lim_{x\to+\infty}\left[\frac{8\cdot x+19}{2\cdot x+\sqrt{4\cdot x^2-8\cdot x+19}}\right]=\lim_{x\to+\infty}\frac{8+\frac{19}{x}}{2+\sqrt{4-8\cdot\frac{1}{x}+\frac{19}{x^2}}}=\frac{8+0}{2+\sqrt{4-0+0}}=2.$$

8. $L=\lim_{x\to-\infty}\left(x+\sqrt{x^2-x+1}\right)=\lim_{x\to-\infty}\left[\left(x+\sqrt{x^2-x+1}\right)\cdot\frac{\left(x-\sqrt{x^2-x+1}\right)}{\left(x-\sqrt{x^2-x+1}\right)}\right]=\lim_{x\to-\infty}\left[\frac{x^2-\left(\sqrt{x^2-x+1}\right)^2}{x-\sqrt{x^2-x+1}}\right]=$

$$= \lim_{x\to-\infty}\left[\frac{x^2-(x^2-x+1)}{x-\sqrt{x^2-x+1}}\right] = \lim_{x\to-\infty}\left(\frac{x-1}{x-\sqrt{x^2-x+1}}\right) = \lim_{x\to-\infty}\left(\frac{\frac{x-1}{x}}{\frac{x-\sqrt{x^2-x+1}}{x}}\right) = \lim_{x\to-\infty}\left(\frac{1-\frac{1}{x}}{1-\frac{\sqrt{x^2-x+1}}{x}}\right).$$

Ovdje **ne smijemo** napisati $\frac{\sqrt{x^2-x+1}}{x} = \frac{\sqrt{x^2-x+1}}{\sqrt{x^2}} = \sqrt{\frac{x^2-x+1}{x^2}}$. Naime, znamo da $x\to-\infty$, pa je x strogo negativan realan broj, a za takve brojeve **ne vrijedi** jednakost $x=\sqrt{x^2}$.

Da bismo mogli izračunati $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2-x+1}}{x}$ moramo uvesti zamjenu $t=-x$. Odatle je $x=-t$. Očito, kad $x\to-\infty$, onda $t\to+\infty$. Tako sada imamo:

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-x+1}}{x} = \lim_{t\to+\infty}\frac{\sqrt{(-t)^2-(-t)+1}}{(-t)} = \lim_{t\to+\infty}\left(-\frac{\sqrt{t^2+t+1}}{t}\right).$$

Sad $t\to+\infty$, pa je t strogo pozitivan realan broj. Za takve brojeve vrijedi jednakost $t=\sqrt{t^2}$. Stoga dobivamo:

$$\lim_{t\to+\infty}\left(-\frac{\sqrt{t^2+t+1}}{t}\right) = \lim_{t\to+\infty}\left(-\sqrt{\frac{t^2+t+1}{t^2}}\right) = \lim_{t\to+\infty}\left(-\sqrt{1+\frac{1}{t}+\frac{1}{t^2}}\right) = -\sqrt{1+0+0} = -1.$$

Tako konačno imamo: $L = \lim\limits_{x\to-\infty}\left(\frac{1-\frac{1}{x}}{1-\frac{\sqrt{x^2-x+1}}{x}}\right) = \frac{1-0}{1-(-1)} = \frac{1}{2}.$

9. *Uputa*: $\frac{(x+1)^2+(x-1)^2}{(1-x)\cdot(x+1)} = \frac{2\cdot x^2+2}{1-x^2} = \frac{\frac{2\cdot x^2+2}{x^2}}{\frac{1-x^2}{x^2}} = \frac{2+\frac{2}{x^2}}{\frac{1}{x^2}-1} \to \frac{2+0}{0-1} = -2$ kad $x\to-\infty$.

10. Rješenje analogno kao u Zadatku 9.

11. $L = \lim\limits_{x\to 0}\left[(1+2\cdot x)^{\frac{1}{3\cdot x}}\right] = \lim\limits_{x\to 0}\left\{\left[(1+2\cdot x)^{\frac{1}{x}}\right]^{\frac{1}{3}}\right\} = \left[\lim\limits_{x\to 0}(1+2\cdot x)^{\frac{1}{x}}\right]^{\frac{1}{3}} = \left(e^2\right)^{\frac{1}{3}} = e^{\frac{2}{3}} = \sqrt[3]{e^2}.$

12. Primijetimo da vrijedi jednakost $\left(1+\frac{1}{x^2}\right)^{(2-x)\cdot(x+2)} = \left(1+\frac{1}{x^2}\right)^{4-x^2}$. Uvedimo zamjenu $t=x^2$. Očito, kad $x\to+\infty$, onda $t\to+\infty$. Stoga dobivamo:

$$L = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{4-t} = \lim_{t\to+\infty}\frac{\left(1+\frac{1}{t}\right)^4}{\left(1+\frac{1}{t}\right)^t} = \frac{\lim\limits_{t\to+\infty}\left(1+\frac{1}{t}\right)^4}{\lim\limits_{t\to+\infty}\left(1+\frac{1}{t}\right)^t} = \frac{(1+0)^4}{e^1} = \frac{1}{e} = e^{-1}.$$