1.5. INTEGRIRANJE RACIONALNIH FUNKCIJA

1. Odredite sljedeće neodređene integrale:

a) $\int \frac{\mathrm{d}x}{x^2+2\cdot x-3}$;

b) $\int \frac{20}{4\cdot t^2-16\cdot t-9}\cdot \mathrm{d}t$;

c) $\int \frac{u}{u^2+u+1}\cdot \mathrm{d}u$;

d) $\int \frac{w-1}{w^2-w+1}\cdot \mathrm{d}w$.

2. Metodom neodređenih koeficijenata odredite sljedeće neodređene integrale:

a) $\int \frac{\mathrm{d}x}{x^3-2\cdot x^2+x}$;

b) $\int \frac{42}{y^3-8\cdot y^2+7\cdot y}\cdot \mathrm{d}y$;

c) $\int \frac{6}{t^3+1}\cdot \mathrm{d}t$;

d) $\int \frac{u^2+2\cdot u+5}{(u-1)\cdot(u^2+3\cdot u+4)}\cdot \mathrm{d}u$.

3. Odredite sljedeće neodređene integrale nepravih racionalnih funkcija:

a) $\int \frac{x^2-5\cdot x+9}{x^2-5\cdot x+6}\cdot \mathrm{d}x$;

b) $\int \frac{y^3+y+1}{y\cdot(y^2+1)}\cdot \mathrm{d}y$.

1.6. INTEGRIRANJE IRACIONALNIH FUNKCIJA

1. Odredite sljedeće neodređene integrale:

a) $\int \sqrt{x^2-x-2}\cdot \mathrm{d}x$;

b) $\int \sqrt{4\cdot y^2+8\cdot y+5}\cdot \mathrm{d}y$;

c) $\int \sqrt{7+6\cdot t-t^2}\cdot \mathrm{d}t$;

d) $\int \sqrt{8-6\cdot u-9\cdot u^2}\cdot \mathrm{d}u$.

2. Odredite sljedeće neodređene integrale:

a) $\int \frac{x}{\sqrt{4\cdot x^2-4\cdot x-3}}\cdot \mathrm{d}x$;

b) $\int \frac{1-y}{\sqrt{y-y^2}}\cdot \mathrm{d}y$.

RJEŠENJA ZADATAKA

Napomena: U svim rješenjima zadataka $C \in \mathbb{R}$ je konstanta.

1.5. Integriranje racionalnih funkcija

1.

a) $\int \frac{\mathrm{d}x}{x^2+2\cdot x-3} = \int \frac{\mathrm{d}x}{(x+1)^2-4} = \frac{1}{2}\cdot \ln\left|\frac{x-1}{x+3}\right| + C;$

b) $\int \frac{20}{4\cdot t^2-16\cdot t-9}\cdot \mathrm{d}t = 5\cdot \int \frac{\mathrm{d}t}{t^2-4\cdot t-\frac{9}{4}} = 5\cdot \int \frac{\mathrm{d}t}{(t-2)^2-\frac{25}{4}} = \ln\left|\frac{2\cdot t-9}{2\cdot t+1}\right| + C;$

c) $u = \check{S}\cdot(2\cdot u+1)+\acute{C} \Rightarrow (\check{S},\acute{C}) = \left(\frac{1}{2},-\frac{1}{2}\right) \Rightarrow \int \frac{u\cdot \mathrm{d}u}{u^2+u+1} = \frac{1}{2}\cdot \ln(u^2+u+1) - \frac{1}{2}\cdot \int \frac{\mathrm{d}u}{u^2+u+1} =$

$= \frac{1}{2}\cdot \ln(u^2+u+1) - \frac{1}{2}\cdot \int \frac{\mathrm{d}u}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}\cdot \ln(u^2+u+1) - \frac{\sqrt{3}}{3}\cdot \operatorname{arctg}\left(\frac{2\cdot u+1}{\sqrt{3}}\right) + C;$

d) $w-1 = \check{S}\cdot(2\cdot w-1)+\acute{C} \Rightarrow (\check{S},\acute{C}) = \left(\frac{1}{2},-\frac{1}{2}\right) \Rightarrow \int \frac{w-1}{w^2-w+1}\cdot \mathrm{d}w = \frac{1}{2}\cdot \ln(w^2-w+1) - \frac{1}{2}\cdot \int \frac{\mathrm{d}w}{w^2-w+1} =$

$= \frac{1}{2}\cdot \ln(w^2+w+1) - \frac{1}{2}\cdot \int \frac{\mathrm{d}w}{\left(w-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}\cdot \ln(w^2-w+1) - \frac{\sqrt{3}}{3}\cdot \operatorname{arctg}\left(\frac{2\cdot w-1}{\sqrt{3}}\right) + C.$

2.

a) $x^3-2\cdot x^2+x = x\cdot(x-1)^2 \Rightarrow \frac{1}{x^3-2\cdot x^2+x} = \frac{A_1}{x}+\frac{A_2}{x-1}+\frac{A_3}{(x-1)^2} \Rightarrow (A_1,A_2,A_3) = (1,-1,\ 1)$

$\Rightarrow \int \frac{\mathrm{d}x}{x^3-2\cdot x^2+x} = \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x-1} + \int \frac{\mathrm{d}x}{(x-1)^2} = \ln|x| - \ln|x-1| + \frac{1}{x-1} + C;$

b) $y^3-8\cdot y^2+7\cdot y = y\cdot(y-1)\cdot(y-7) \Rightarrow \frac{42}{y^3-8\cdot y^2+7\cdot y} = \frac{A_1}{y}+\frac{A_2}{y-1}+\frac{A_3}{y-7} \Rightarrow (A_1,A_2,A_3) = (6,-7,\ 1) \Rightarrow$

$\int \frac{\mathrm{d}y}{y^3-8\cdot y^2+7\cdot y} = 6\cdot \int \frac{\mathrm{d}y}{x} - 7\cdot \int \frac{\mathrm{d}y}{x-1} + \int \frac{\mathrm{d}y}{x-7} = 6\cdot \ln|y| - 7\cdot \ln|y-1| + \ln|y-7| + C;$

c) $t^3+1 = (t+1)\cdot(t^2-t+1) \Rightarrow \frac{6}{t^3+1} = \frac{A_1}{t+1}+\frac{A_2\cdot t+A_3}{t^2-t+1} \Rightarrow (A_1,A_2,A_3) = (2,-2,\ 4) \Rightarrow \int \frac{6}{t^3+1}\cdot \mathrm{d}t =$

$= 2\cdot \int \frac{\mathrm{d}t}{t+1} - 2\cdot \int \frac{t-2}{t^2-t+1}\cdot \mathrm{d}t = 2\cdot \ln|t+1| + 2\cdot \sqrt{3}\cdot \operatorname{arctg}\left(\frac{2\cdot t+1}{\sqrt{3}}\right) - \ln(t^2-t+1) + C;$

d) $\frac{u^2+2\cdot u+5}{(u-1)\cdot(u^2+3\cdot u+4)} = \frac{A_1}{u-1}+\frac{A_2\cdot u+A_3}{u^2+3\cdot u+4} \Rightarrow (A_1,A_2,A_3) = (1,\ 0,-1) \Rightarrow \int \frac{u^2+2\cdot u+5}{(u-1)\cdot(u^2+3\cdot u+4)}\cdot \mathrm{d}u =$

$= \int \frac{\mathrm{d}u}{u-1} - \int \frac{\mathrm{d}x}{u^2+3\cdot u+4} = \ln|u-1| - \frac{2}{7}\cdot \sqrt{7}\cdot \operatorname{arctg}\left(\frac{2\cdot u+3}{\sqrt{7}}\right) + C.$

3.

a) $\int \frac{x^2-5\cdot x+9}{x^2-5\cdot x+6}\cdot \mathrm{d}x = \int\left(1+\frac{3}{x^2-5\cdot x+6}\right)\cdot \mathrm{d}x = \int \mathrm{d}x + 3\cdot\int \frac{\mathrm{d}x}{x^2-5\cdot x+6} = \int \mathrm{d}x + 3\cdot\int \frac{\mathrm{d}x}{x-3} - 3\cdot\int \frac{\mathrm{d}x}{x-2} =$

$= x + 3\cdot\ln|x-3| - 3\cdot\ln|x-2| + C;$

b) $\int \frac{y^3+y+1}{y\cdot(y^2+1)}\cdot \mathrm{d}y = \int \mathrm{d}y + \int \frac{\mathrm{d}y}{y\cdot(y^2+1)} = \int \mathrm{d}y + \int \frac{\mathrm{d}y}{y} - \int \frac{y}{y^2+1}\cdot \mathrm{d}y = y + \ln|y| - \frac{1}{2}\cdot\ln(y^2+1) + C.$

1.6. Integriranje iracionalnih funkcija

1.

a) $\int \sqrt{x^2-x-2}\cdot \mathrm{d}x = \int \sqrt{\left(x-\frac{1}{2}\right)^2 - \frac{9}{4}}\cdot \mathrm{d}x = \frac{2\cdot x-1}{4}\cdot\sqrt{x^2-x-2} - \frac{9}{8}\cdot\ln\left(x-\frac{1}{2}+\sqrt{x^2-x-2}\right) + C;$

b) $\int \sqrt{4\cdot y^2+8\cdot y+5}\cdot \mathrm{d}y = 2\cdot\int \sqrt{(y+1)^2+\frac{1}{4}}\cdot \mathrm{d}y = 2\cdot\left[\frac{y+1}{2}\cdot\sqrt{(y+1)^2+\frac{1}{4}} + \frac{1}{4}\cdot\ln\left(y+1+\sqrt{(y+1)^2+\frac{1}{4}}\right)\right] =$

$= \frac{y+1}{2}\cdot\sqrt{4\cdot y^2+8\cdot y+5} + \frac{1}{2}\cdot\ln\left(y+1+\frac{1}{2}\cdot\sqrt{4\cdot y^2+8\cdot y+5}\right) + C;$

c) $\int \sqrt{7+6\cdot t-t^2}\cdot \mathrm{d}t = \int \sqrt{16-(3-t)^2}\cdot \mathrm{d}t = \frac{t-3}{2}\cdot\sqrt{7+6\cdot t-t^2} + 8\cdot\arcsin\left(\frac{t-3}{4}\right) + C;$

d) $\int \sqrt{8-6\cdot u-9\cdot u^2}\cdot \mathrm{d}u = \int \sqrt{9-(1+3\cdot u)^2}\cdot \mathrm{d}u = \frac{3\cdot u+1}{6}\cdot\sqrt{8-6\cdot u-9\cdot u^2} + \frac{3}{2}\cdot\arcsin\left(\frac{3\cdot u+1}{3}\right) + C.$

2.

a) $x = \check{S}\cdot(8\cdot x-4) + \acute{C} \Rightarrow (\check{S},\acute{C}) = \left(\frac{1}{8},\ \frac{1}{2}\right), \Rightarrow \int \frac{x}{\sqrt{4\cdot x^2-4\cdot x-3}}\cdot \mathrm{d}x = \frac{1}{8}\cdot\int \frac{8\cdot x-4}{\sqrt{4\cdot x^2-4\cdot x-3}}\cdot \mathrm{d}x +$

$+ \frac{1}{2}\cdot\int \frac{\mathrm{d}x}{\sqrt{4\cdot x^2-4\cdot x-3}} = \frac{1}{4}\cdot\sqrt{4x^2-4x-3} + \frac{1}{2}\cdot\int \frac{\mathrm{d}x}{\sqrt{(2\cdot x-1)^2-4}} = \frac{1}{4}\cdot\sqrt{4\cdot x^2-4\cdot x-3} +$

$+ \frac{1}{4}\cdot\ln\left|2\cdot x-1+\sqrt{4\cdot x^2+4\cdot x-3}\right| + C;$

b) $1-y = \check{S}\cdot(1-2\cdot y) + \acute{C} \Rightarrow (\check{S},\acute{C}) = \left(\frac{1}{2},\ \frac{1}{2}\right) \Rightarrow \int \frac{1-y}{\sqrt{y-y^2}}\cdot \mathrm{d}y = \frac{1}{2}\cdot\int \frac{1-2\cdot y}{\sqrt{y-y^2}}\cdot \mathrm{d}y + \frac{1}{2}\cdot\int \frac{\mathrm{d}y}{\sqrt{y-y^2}} =$

$= \sqrt{y-y^2} + \frac{1}{2}\cdot\int \frac{\mathrm{d}y}{\sqrt{\frac{1}{4}-\left(\frac{1}{2}-y\right)^2}} = \sqrt{y-y^2} + \frac{1}{2}\cdot\arcsin(2\cdot y-1) + C.$