1. Odredite integralni zbroj S_n za funkciju $f(x) = x$ definiranu na segmentu $[0, 1]$ tako da pripadnim diobenim točkama taj segment bude podijeljen na jednake dijelove. Izračunajte $\lim\limits_{n\to\infty} S_n$ i objasnite dobiveni rezultat.

2. Izračunajte sljedeće određene integrale:

a) $\int\limits_0^{\frac{\pi^2}{4}} \cos\sqrt{x} \cdot \mathrm{d}x;$

b) $\int\limits_0^{\frac{\pi}{2}} \sin(2\cdot y)\cdot e^{\sin y}\cdot \mathrm{d}y.$

3. Izračunajte prosječnu vrijednost funkcije:

a) $f_1(x) = \sin x$ na segmentu $[0,\ 2\cdot\pi]$;

b) $f_2(x) = \lfloor x \rfloor$ na segmentu $[-2\cdot\pi,\ 2\cdot\pi]$;

c) $g(y) = \arcsin y$ na njezinu prirodnu području definicije;

d) $h(t) = \sqrt{9-t^2}$ na njezinu prirodnu području definicije;

4. Izračunajte površinu ravninskoga lika omeđenoga krivuljama:

a) $y = x^2 - 5\cdot x + 6$ i $y = 0$;

b) $y = \ln x$, $y = 0$, $x = 1$ i $x = e$;

c) $y = \arcsin x$, $y = x$, $y = 0$, $x = 0$ i $x = 1$;

d) $y = 2\cdot x - x^2$ i $x + y = 0$.

5. Izračunajte površinu ravninskoga lika kojega zatvaraju krivulja $y = -\dfrac{2}{x^2+1}$, normala na tu krivulju povučena u točki krivulje $T = (x_T > 0,\ -1)$ i os ordinata.

6. Izračunajte duljinu luka krivulje:

a) $y = \dfrac{2}{3}\cdot(x-1)\cdot\sqrt{x-1}$ iznad segmenta [1, 4];

b) $y = \operatorname{ch} x$ iznad segmenta [0, ln 5];

c) $y = 2\cdot\left[\sqrt{e^x-1} - \operatorname{arctg}\left(\sqrt{e^x-1}\right)\right]$ iznad segmenta [0, 2 · ln 2].

7. Izračunajte obujam rotacijskoga tijela nastaloga vrtnjom krivocrtnoga trapeza omeđenoga krivuljama:

a) $y = \sin x$, $y = 0$, $x = 0$ i $x = \pi$ oko osi apscisa;

b) $y = \ln x$, $y = 0$, $x = 1$ i $x = e$ oko osi ordinata;

c) $y = \operatorname{tg} x$, $y = 0$, $x = \pi$ i $x = \dfrac{5}{4}\cdot\pi$ oko osi apscisa;

d) $y = e^{2\cdot x}$, $y = 0$, $x = 0$ i $x = 1$ oko osi ordinata.

RJEŠENJA ZADATAKA

1. Diobene točke su $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., \frac{n-1}{n}$. Stavimo li $x_0 = 0$, $x_n = 1$, onda za svaki $i = 1, \ldots, n - 1$ vrijedi $x_i = \frac{i}{n}$, $x_{i+1} - x_i = \frac{1}{n}$, te $f(x_i) = x_i = \frac{i}{n}$. Stoga je:

$$S_n = \sum_{i=1}^{n-1} f(x_i) \cdot (x_{i+1} - x_i) = \sum_{i=1}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n-1} i = \frac{1}{n^2} \cdot \frac{(n-1) \cdot n}{2} = \frac{n^2 - n}{2 \cdot n^2} = \frac{1}{2} - \frac{1}{2 \cdot n}.$$

Odatle slijedi:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2 \cdot n} \right) = \frac{1}{2}.$$

Prema tome, površina krivocrtnoga trapeza omeđenoga krivuljama $y = x$, $x = 0$ i $x = 1$ iznosi $P = \frac{1}{2}$ kv. jed. No, taj je lik zapravo jednakokračan pravokutan trokut kojemu su duljine kateta jednake $a = 1$, pa je njegova površina jednaka $P = \frac{a^2}{2} = \frac{1}{2}$ kv. jed.

2.

a) $$\int \cos\sqrt{x} \cdot \mathrm{d}x = \left\{ \begin{matrix} t := \sqrt{x}, \\ x = t^2, \\ \mathrm{d}x = 2 \cdot t \cdot \mathrm{d}t \end{matrix} \right\} = 2 \cdot \int t \cdot \cos t \cdot \mathrm{d}t = \left\{ \begin{matrix} u = t & v = \sin t \\ \mathrm{d}u = 1 \cdot \mathrm{d}t & \mathrm{d}v = \cos t \cdot \mathrm{d}t \end{matrix} \right\} = 2 \cdot \left(t \cdot \sin t - \int \sin t \cdot dt \right) =$$

$$= 2 \cdot (t \cdot \sin t + \cos t) = 2 \cdot \left(\sqrt{x} \cdot \sin\sqrt{x} + \cos\sqrt{x} \right) + C \Rightarrow$$

$$F(x) = 2 \cdot \left(\sqrt{x} \cdot \sin\sqrt{x} + \cos\sqrt{x} \right) \Rightarrow$$

$$\int_0^{\frac{\pi^2}{4}} \cos\sqrt{x} \cdot \mathrm{d}x = F\left(\frac{\pi^2}{4} \right) - F(0) = \pi - 2;$$

b) $$\int \sin(2 \cdot y) \cdot e^{\sin y} \cdot \mathrm{d}y = 2 \cdot \int \sin y \cdot \cos y \cdot e^{\sin y} \cdot \mathrm{d}y = \left\{ \begin{matrix} t := \sin y \\ \mathrm{d}x = \cos y \cdot \mathrm{d}y \end{matrix} \right\} = 2 \cdot \int t \cdot e^t \cdot \mathrm{d}t = \left\{ \begin{matrix} u = t & v = e^t \\ \mathrm{d}u = 1 \cdot \mathrm{d}t & \mathrm{d}v = e^t \cdot \mathrm{d}t \end{matrix} \right\} =$$

$$= 2 \cdot \left(t \cdot e^t - \int e^t dt \right) = 2 \cdot \left(t \cdot e^t - e^t \right) = 2 \cdot e^t \cdot (t - 1) = 2 \cdot e^{\sin y} \cdot (\sin y - 1) + C \Rightarrow$$

$$F(y) = 2 \cdot e^{\sin y} \cdot (\sin y - 1) \Rightarrow$$

$$\int_0^{\frac{\pi}{2}} \sin(2 \cdot y) \cdot e^{\sin y} \cdot \mathrm{d}y = F\left(\frac{\pi}{2} \right) - F(0) = 2.$$

3.

a) $\overline{f}_{1[0,2\cdot\pi]}=\frac{1}{2\cdot\pi-0}\cdot\int\limits_0^{2\cdot\pi}\sin x\cdot\mathrm{d}x=\frac{1}{2\cdot\pi}\cdot[\cos 0-\cos(2\cdot\pi)]=0;$

b) $\overline{f}_{2[-2\cdot\pi,2\cdot\pi]}=\frac{1}{2\cdot\pi-(-2\cdot\pi)}\cdot\int\limits_{-2\cdot\pi}^{2\cdot\pi}\lfloor x\rfloor\cdot\mathrm{d}x=\frac{1}{4\cdot\pi}\cdot\left(\int\limits_{-2\cdot\pi}^{-6}\lfloor x\rfloor\cdot\mathrm{d}x+\sum_{k=-6}^{5}\int\limits_k^{k+1}\lfloor x\rfloor\cdot\mathrm{d}x+\int\limits_6^{2\cdot\pi}\lfloor x\rfloor\cdot\mathrm{d}x\right)=$

$=\frac{1}{4\cdot\pi}\cdot\left(\int\limits_{-2\cdot\pi}^{-6}-7\cdot\mathrm{d}x+\sum_{k=-6}^{5}\int\limits_k^{k+1}k\cdot\mathrm{d}x+\int\limits_6^{2\cdot\pi}6\cdot\mathrm{d}x\right)=\frac{1}{4\cdot\pi}\cdot\left[(-7)\cdot\int\limits_{-2\cdot\pi}^{-6}1\cdot\mathrm{d}x+\sum_{k=-6}^{5}k\cdot\int\limits_k^{k+1}1\cdot\mathrm{d}x+6\cdot\int\limits_6^{2\cdot\pi}1\cdot\mathrm{d}x\right]=$

$=\frac{1}{4\cdot\pi}\cdot\left[(-7)\cdot(-6+2\cdot\pi)+\sum_{k=-6}^{5}k\cdot 1+7\cdot(2\cdot\pi-6)\right]=$

$=\frac{1}{4\cdot\pi}\cdot\left[(-7)\cdot(-6+2\cdot\pi)+\frac{12}{2}\cdot(-6+5)+6\cdot(2\cdot\pi-6)\right]=\frac{1}{4\cdot\pi}\cdot(-2\cdot\pi)=-\frac{1}{2};$

c) $\overline{g}_{[-1,1]}=\frac{1}{1-(-1)}\cdot\int\limits_{-1}^{1}\arcsin y\cdot\mathrm{d}y=\frac{1}{2}\cdot\left[y\cdot\arcsin y+\sqrt{1-y^2}\right]_{-1}^{1}=0;$

d) $\overline{h}_{[-3,3]}=\frac{1}{3-(-3)}\cdot\int\limits_{-3}^{3}\sqrt{9-t^2}\cdot\mathrm{d}t=\frac{1}{6}\cdot\frac{9}{2}\cdot[\arcsin 1-\arcsin(-1)]=\frac{3}{4}\cdot\pi.$

4.

a) $P=\left|\int\limits_2^3(x^2-5\cdot x+6)\cdot\mathrm{d}x\right|=\left|-\frac{1}{6}\right|=\frac{1}{6}$ kv. jed.;

b) $P=\int\limits_1^e\ln x\cdot\mathrm{d}x=1$ kv. jed.;

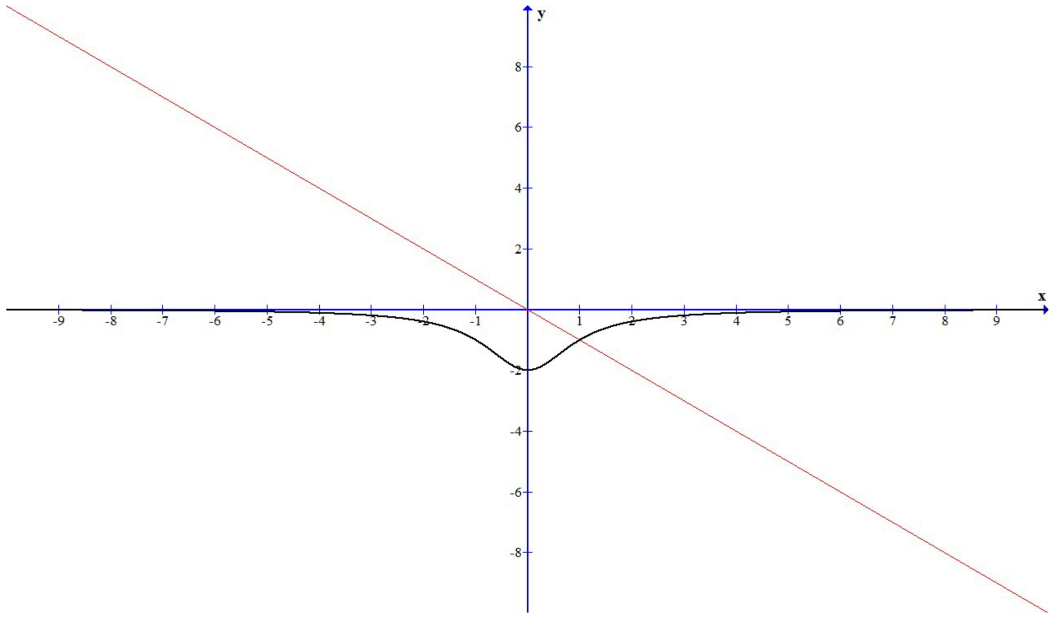
c) $P=\int\limits_0^1(\arcsin x-x)\cdot\mathrm{d}x=\int\limits_0^1\arcsin x\cdot\mathrm{d}x-\int\limits_0^1 x\cdot\mathrm{d}x=\frac{\pi-3}{2}$ kv. jed.;

d) $P=\int\limits_0^2(2\cdot x-x^2)\cdot\mathrm{d}x+\left|\int\limits_0^3-x\cdot\mathrm{d}x\right|-\left|\int\limits_2^3(2\cdot x-x^2)\cdot\mathrm{d}x\right|=\frac{9}{2}$ kv. jed.

5. Uvrstimo $y=-1$ u jednadžbu krivulje, pa iz jednadžbe $-\frac{2}{x^2+1}=-1$ i uvjeta $x>0$ slijedi $x=1$. Dakle, $T=(1,-1)$.

Deriviranjem izraza $y=-\frac{2}{x^2+1}$ dobijemo $y'=\frac{4\cdot x}{(x^2+1)^2}$, pa je koeficijent smjera normale $k_n=-\frac{1}{y'(1)}=-1$. Stoga je jednadžba normale $y=-x$.

Nacrtamo pripadnu sliku (vidjeti Sliku 1).



Slika 1.

Tako lagano slijedi:

$$P=\left|\int_0^1\left(-\frac{2}{x^2+1}-x\right)\cdot\mathrm{d}x\right|=\frac{\pi-1}{2}\text{ kv. jed.}$$

6.

a) $l=\int_1^4\sqrt{1+(x-1)}\cdot\mathrm{d}x=\int_1^4\sqrt{x}\cdot\mathrm{d}x=\frac{14}{3}$ jed.;

b) $l=\int_0^{\ln 5}\sqrt{1+(\operatorname{sh}x)^2}\cdot\mathrm{d}x=\int_0^{\ln 5}\operatorname{ch}x\cdot\mathrm{d}x=\frac{12}{5}$ jed.;

c) $l=\int_0^{2\cdot\ln 2}\sqrt{1+(e^x-1)}\cdot\mathrm{d}x=\int_0^{2\cdot\ln 2}e^{\frac{x}{2}}\cdot\mathrm{d}x=2$ jed.

7.

a) $V=\pi\cdot\int_0^{\pi}\sin^2 x\cdot\mathrm{d}x=\pi\cdot\int_0^{\pi}\left[\frac{1-\cos(2\cdot x)}{2}\right]\cdot\mathrm{d}x=\frac{\pi^2}{2}$ kub. jed.;

b) $V=2\cdot\pi\cdot\int_1^{e}x\cdot\ln x\cdot\mathrm{d}x=\frac{\pi}{2}\cdot(e^2+1)$ kub. jed.;

c) $V=\pi\cdot\int_{\pi}^{\frac{5\cdot\pi}{4}}\operatorname{tg}^2 x\cdot\mathrm{d}x=\pi\cdot\int_{\pi}^{\frac{5\cdot\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)\cdot\mathrm{d}x=\pi-\frac{\pi^2}{4}$ kub. jed.;

d) $V=2\cdot\pi\cdot\int x\cdot e^{2\cdot x}\cdot\mathrm{d}x=\frac{\pi}{2}\cdot(e^2+1)$ kub. jed.