1. Riješite sljedeće jednadžbe:

a) $y'' - 3\cdot y' + 2\cdot y = 0;$

b) $y'' + 5\cdot y' + 6\cdot y = 0;$

c) $y'' - 12\cdot y' + 36\cdot y = 0;$

d) $y'' + 10\cdot y' + 25\cdot y = 0;$

e) $y'' + 4\cdot y = 0;$

f) $y'' + 2\cdot y' + 2\cdot y = 0;$

g) $y'' - 4\cdot y' + 13\cdot y = 0.$

2. Riješite sljedeće jednadžbe:

a) $y'' - 4\cdot y' + 4\cdot y = 4\cdot x^2 - 12\cdot x + 2;$

b) $y'' - 8\cdot y' + 7\cdot y = 14;$

c) $y'' + 2\cdot y' + y = 9\cdot e^{2\cdot x};$

d) $y'' + y' - 2\cdot y = 8\cdot \cos(2\cdot x) - 4\cdot \sin(2\cdot x);$

e) $y'' - 3\cdot y' = (3\cdot x - 2)\cdot \sin x - (x+3)\cdot \cos x;$

f) $y'' + y' = 6\cdot x + 2\cdot e^x;$

g) $y'' + y' = 20\cdot \sin^2 x;$

h) $y'' - y = 4\cdot x\cdot e^x;$

i) $y'' - 4\cdot y = 8\cdot e^{2\cdot x}\cdot [2\cdot \cos(2\cdot x) - \sin(2\cdot x)].$

3. Riješite sljedeće Cauchyjeve probleme:

a) $\begin{cases} y'' + y' - 6\cdot y = (10\cdot x - 3)\cdot e^{2\cdot x}, \\ y(0) = 1, \\ y'(0) = -4. \end{cases}$

b) $\begin{cases} y'' - 2\cdot y' + 10\cdot y = 37\cdot \sin(3\cdot x) + 18\cdot e^x, \\ y(0) = 10, \\ y'(0) = -5. \end{cases}$

RJEŠENJA ZADATAKA

Napomena: Ako se ne istakne drugačije, $C_1, C_2 \in \mathbb{R}$ su konstante.

1.

a) Karakteristična jednadžba (u daljnjem tekstu: K.J.) je $k^2 - 3 \cdot k + 2 = 0$. Njezina rješenja su $k_1 = 1$ i $k_2 = 2$. Zbog toga je $y = C_1 \cdot e^x + C_2 \cdot e^{2 \cdot x}$.

b) K.J.: $k^2 + 5 \cdot k + 6 = 0 \Rightarrow k_1 = -2, k_2 = -3 \Rightarrow y = C_1 \cdot e^{-2 \cdot x} + C_2 \cdot e^{-3 \cdot x}$.

c) K.J.: $k^2 - 12 \cdot k + 36 = 0 \Rightarrow k_1 = k_2 = 6 \Rightarrow y = (C_1 \cdot x + C_2) \cdot e^{6 \cdot x}$.

d) K.J.: $k^2 + 10 \cdot k + 25 = 0 \Rightarrow k_1 = k_2 = -5 \Rightarrow y = (C_1 \cdot x + C_2) \cdot e^{-5 \cdot x}$.

e) K.J.: $k^2 + 4 = 0 \Rightarrow k_1 = 2 \cdot i \Rightarrow y = C_1 \cdot \cos(2 \cdot x) + C_2 \cdot \sin(2 \cdot x)$.

f) K.J.: $k^2 + 2 \cdot k + 2 = 0 \Rightarrow k_1 = -1 + i \Rightarrow y = (C_1 \cdot \cos x + C_2 \cdot \sin x) \cdot e^{-x}$.

g) K.J.: $k^2 - 4 \cdot k + 13 = 0 \Rightarrow k_1 = 2 + 3 \cdot i \Rightarrow y = \left[C_1 \cdot \cos(3 \cdot x) + C_2 \cdot \sin(3 \cdot x)\right] \cdot e^{2 \cdot x}$.

2.

S y_h označeno je rješenje pripadne homogene obične diferencijalne jednadžbe 2. reda, a s y_p partikularno rješenje zadane jednadžbe.

a) K.J.: $k^2 - 4 \cdot k + 4 = 0 \Rightarrow k_1 = k_2 = 2 \Rightarrow y_h = (C_1 \cdot x + C_2) \cdot e^{2 \cdot x}$;
$y_p = a \cdot x^2 + b \cdot x + c \Rightarrow (a, b, c) = (1, -1, 1) \Rightarrow y_p = x^2 - x + 1$.
Rješenje zadatka je: $y = (C_1 \cdot x + C_2) \cdot e^{2 \cdot x} + x^2 - x + 1$.

b) K.J.: $k^2 - 8 \cdot k + 7 = 0 \Rightarrow k_1 = 1, k_2 = 7 \Rightarrow y_h = C_1 \cdot e^x + C_2 \cdot e^{7 \cdot x}$;
$y_p = a \Rightarrow a = 2 \Rightarrow y_p = 2$.
Rješenje zadatka je: $y = C_1 \cdot e^x + C_2 \cdot e^{7 \cdot x} + 2$.

c) K.J.: $k^2 + 2 \cdot k + 1 = 0 \Rightarrow k_1 = k_2 = -1 \Rightarrow y_h = (C_1 \cdot x + C_2) \cdot e^{-x}$.
$y_p = a \cdot e^{2 \cdot x} \Rightarrow a = 1 \Rightarrow y_p = e^{2 \cdot x}$.
Rješenje zadatka je $y = (C_1 \cdot x + C_2) \cdot e^{-x} + e^{2 \cdot x}$.

d) K.J.: $k^2 + k - 2 = 0 \Rightarrow k_1 = -2, k_2 = 1 \Rightarrow y_h = C_1 \cdot e^{-2x} + C_2 \cdot e^x$;
$y_p = a \cdot \cos(2 \cdot x) + b \cdot \sin(2 \cdot x) \Rightarrow (a, b) = (-1, 1) \Rightarrow y_p = \sin(2 \cdot x) - \cos(2 \cdot x)$..
Rješenje zadatka je: $y = C_1 \cdot e^{-2x} + C_2 \cdot e^x + \sin(2 \cdot x) - \cos(2 \cdot x)$.

e) K.J.: $k^2 - 3 \cdot k = 0 \Rightarrow k_1 = 0, k_2 = 3 \Rightarrow y_h = C_1 \cdot e^{3 \cdot x} + C_2$.
$y_p = (a \cdot x + b) \cdot \cos x + (c \cdot x + d) \cdot \sin x \Rightarrow (a, b, c, d) = (1, 0, 0, 0) \Rightarrow y_p = x \cdot \cos x$.
Rješenje zadatka je: $y = C_1 \cdot e^{3 \cdot x} + x \cdot \cos x + C_2$.

f) K.J.: $k^2 + k = 0 \Rightarrow k_1 = -1, k_2 = 0 \Rightarrow y_h = C_1 \cdot e^{-x} + C_2$;
$(y_p)_1 = a \cdot x^2 + b \cdot x \Rightarrow (a, b) = (3, -6) \Rightarrow (y_p)_1 = 3 \cdot x^2 - 6 \cdot x$

$(y_p)_2 = d \cdot e^x \Rightarrow d = 1 \Rightarrow (y_p)_2 = e^x$.

Rješenje zadatka je: $y = C_1 \cdot e^{-x} + e^x + 3 \cdot x^2 - 6 \cdot x + C_2$.

g) K.J.: $k^2 + k = 0 \Rightarrow k_1 = -1,\ k_2 = 0 \Rightarrow y_h = C_1 \cdot e^{-x} + C_2$;

$y_p = a \cdot x + b \cdot \cos(2 \cdot x) + c \cdot \sin(2 \cdot x) \Rightarrow (a,b,c) = (10, 2, -1) \Rightarrow y_p = 10 \cdot x + 2 \cdot \cos(2 \cdot x) - \sin(2 \cdot x)$.

Rješenje zadatka je: $y = C_1 \cdot e^{-x} + 2 \cdot \cos(2 \cdot x) - \sin(2 \cdot x) + 10 \cdot x + C_2$.

h) K.J.: $k^2 - 1 = 0 \Rightarrow k_1 = -1,\ k_2 = 1 \Rightarrow y_h = C_1 \cdot e^{-x} + C_2 \cdot e^x$;

$y_p = (a \cdot x^2 + b \cdot x) \cdot e^x \Rightarrow (a,b) = (1,-1) \Rightarrow y_p = (x^2 - x) \cdot e^x$.

Rješenje zadatka je: $y = C_1 \cdot e^{-x} + (x^2 - x + C_2) \cdot e^x$.

i) K.J.: $k^2 - 1 = 0 \Rightarrow k_1 = -2,\ k_2 = 2 \Rightarrow y_h = C_1 \cdot e^{-2 \cdot x} + C_2 \cdot e^{2 \cdot x}$;

$y_p = \left[a \cdot \cos(2 \cdot x) + b \cdot \sin(2 \cdot x)\right] \cdot e^{2 \cdot x} \Rightarrow (a,b) = (0,2) \Rightarrow y_p = 2 \cdot \sin(2 \cdot x) \cdot e^{2 \cdot x}$.

Rješenje zadatka je: $y = C_1 \cdot e^{-2 \cdot x} + \left[2 \cdot \sin(2 \cdot x) + C_2\right] \cdot e^{2 \cdot x}$.

3.

S y_h označeno je rješenje pripadne homogene obične diferencijalne jednadžbe 2. reda, s y_p partikularno rješenje zadane jednadžbe. a s y_o opće rješenje zadane jednadžbe.

a) K.J.: $k^2 + k - 6 = 0 \Rightarrow k_1 = -3,\ k_2 = 2 \Rightarrow y_h = C_1 \cdot e^{-3 \cdot x} + C_2 \cdot e^{2 \cdot x}$.

$y_p = (a \cdot x^2 + b \cdot x) \cdot e^{2 \cdot x} \Rightarrow (a,b) = (1,-1) \Rightarrow y_p = (x^2 - x) \cdot e^{2 \cdot x}$.

Stoga je $y_o = C_1 \cdot e^{-3 \cdot x} + (x^2 - x + C_2) \cdot e^{2 \cdot x}$.

$y(0) = 1 \Rightarrow C_1 + C_2 = 1$.

$y'(0) = 4 \Rightarrow -3 \cdot C_1 + 2 \cdot C_2 = -3 \Rightarrow (C_1, C_2) = (1, 0)$.

Rješenje zadatka je: $y = e^{-3 \cdot x} + (x^2 - x) \cdot e^{2 \cdot x}$.

b) K.J.: $k^2 - 2 \cdot k + 10 = 0 \Rightarrow k_1 = 1 + 3 \cdot i \Rightarrow y_h = \left[C_1 \cdot \cos(3 \cdot x) + C_2 \cdot \sin(3 \cdot x)\right] \cdot e^x$;

$(y_p)_1 = a \cdot \cos(3 \cdot x) + b \cdot \sin(3 \cdot x) \Rightarrow (a,b) = (6,1) \Rightarrow (y_p)_1 = 6 \cdot \cos(3 \cdot x) + \sin(3 \cdot x)$;

$(y_p)_2 = d \cdot e^x \Rightarrow d = 2 \Rightarrow (y_p)_2 = 2 \cdot e^x$.

Stoga je $y_o = \left[C_1 \cdot \cos(3 \cdot x) + C_2 \cdot \sin(3 \cdot x) + 2\right] \cdot e^x + 6 \cdot \cos(3 \cdot x) - \sin(3 \cdot x)$.

$y(0) = 10 \Rightarrow C_1 + 8 = 10 \Rightarrow C_1 = 2$.

$y'(0) = -5 \Rightarrow 3 \cdot C_2 + 7 = -5 \Rightarrow C_2 = -4$.

Rješenje zadatka je: $y = 2 \cdot \left[\cos(3 \cdot x) - 2 \cdot \sin(3 \cdot x) + 1\right] \cdot e^x + 6 \cdot \cos(3 \cdot x) - \sin(3 \cdot x)$.