Metodom varijacije konstanti riješite sljedeće jednadžbe:

1. $y'' + y = \operatorname{tg} x.$

2. $y'' + y = \operatorname{ctg} t.$

3. $y'' - 2 \cdot y' + y = \dfrac{e^u}{u}.$

4. $y'' + 2 \cdot y' + y = \dfrac{e^{-w}}{w}.$

5. $y'' + y = \dfrac{1}{\cos z}.$

6. $y'' + y = \dfrac{1}{\sin \alpha}.$

7. $y'' - y = \operatorname{th} \beta.$

RJEŠENJA ZADATAKA

Napomena: U svim rezultatima zadataka su C_1, $C_2 \in \mathbb{R}$ konstante.

1. $k^2+1=0 \Rightarrow k_1=-i, k_2=i \Rightarrow y_h=C_1\cdot\cos x+C_2\cdot\sin x$

$$\Rightarrow \begin{cases} C_1'(x)\cdot\cos x+C_2'(x)\cdot\sin x=0, \\ -C_1'(x)\cdot\sin x+C_2'(x)\cdot\cos x=\operatorname{tg} x \end{cases}$$

$$\Rightarrow (C_1,\ C_2)=\left(\sin x+\ln\left[\operatorname{ctg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]+C_1,\ -\cos x+C_2\right)$$

$$\Rightarrow y=C_1\cdot\cos x+C_2\cdot\sin x+\cos x\cdot\ln\left[\operatorname{ctg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right].$$

2. $k^2+1=0 \Rightarrow k_1=-i, k_2=i \Rightarrow y_h=C_1\cdot\cos t+C_2\cdot\sin t$

$$\Rightarrow \begin{cases} C_1'(t)\cdot\cos t+C_2'(t)\cdot\sin t=0, \\ -C_1'(t)\cdot\sin t+C_2'(t)\cdot\cos t=\operatorname{ctg} t \end{cases}$$

$$\Rightarrow (C_1,\ C_2)=\left(-\sin t+C_1,\ \cos t+\ln\left(\operatorname{tg}\frac{t}{2}\right)+C_2\right)$$

$$\Rightarrow y=C_1\cdot\cos t+C_2\cdot\sin t+\sin t\cdot\ln\left(\operatorname{tg}\frac{t}{2}\right).$$

3. $k^2-2\cdot k+1=0 \Rightarrow k_1=k_2=1 \Rightarrow y_h=C_1\cdot u\cdot e^u+C_2\cdot e^u$

$$\Rightarrow \begin{cases} C_1'(u)\cdot u\cdot e^u+C_2'(u)\cdot e^u=0, \\ C_1'(u)\cdot(u+1)\cdot e^u+C_2'(u)\cdot e^u=\dfrac{e^u}{u} \end{cases}$$

$$\Rightarrow (C_1, C_2)=(\ln u+C_1,\ -u+C_2)$$

$$\Rightarrow y=C_1\cdot u\cdot e^u+C_2\cdot e^u+u\cdot\ln u\cdot e^u.$$

4. $k^2+2\cdot k+1=0 \Rightarrow k_1=k_2=-1 \Rightarrow y_h=C_1\cdot w\cdot e^{-w}+C_2\cdot e^{-w}$

$$\Rightarrow \begin{cases} C_1'(w)\cdot w\cdot e^{-w}+C_2'(w)\cdot e^{-w}=0, \\ C_1'(w)\cdot(1-w)\cdot e^{-w}-C_2'(w)\cdot e^{-w}=\dfrac{e^{-w}}{w} \end{cases}$$

$$\Rightarrow (C_1,\ C_2)=(\ln w+C_1,\ -w+C_2)$$

$$\Rightarrow y=C_1\cdot w\cdot e^{-w}+C_2\cdot e^{-w}+w\cdot\ln w\cdot e^{-w}.$$

5. $k^2+1=0 \Rightarrow k_1=-i, k_2=i \Rightarrow y_h=C_1\cdot\cos z+C_2\cdot\sin z$

$$\Rightarrow \begin{cases} C_1'(z)\cdot\cos z+C_2'(z)\cdot\sin z=0, \\ -C_1'(z)\cdot\sin z+C_2'(z)\cdot\cos z=\dfrac{1}{\cos z} \end{cases}$$

$$\Rightarrow \left(C_1,\ C_2\right) = \left(\ln(\cos z) + C_1,\ \mathrm{z} + C_2\right)$$

$$\Rightarrow y = C_1 \cdot \cos z + C_2 \cdot \sin z + z \cdot \sin z + \cos z \cdot \ln(\cos z).$$

6. $k^2 + 1 = 0 \Rightarrow k_1 = -i, k_2 = i \Rightarrow y_h = C_1 \cdot \cos\alpha + C_2 \cdot \sin\alpha$

$$\Rightarrow \begin{cases} C_1^{'}(\alpha) \cdot \cos\alpha + C_2^{'}(\alpha) \cdot \sin\alpha = 0, \\ -C_1^{'}(\alpha) \cdot \sin\alpha + C_2^{'}(\alpha) \cdot \cos\alpha = \dfrac{1}{\sin\alpha} \end{cases}$$

$$\Rightarrow \left(C_1,\ C_2\right) = \left(-\alpha + C_1,\ \ln(\sin\alpha) + C_2\right)$$

$$\Rightarrow y = C_1 \cdot \cos\alpha + C_2 \cdot \sin\alpha - \alpha \cdot \cos\alpha + \sin\alpha \cdot \ln(\sin\alpha).$$

7. $k^2 - 1 = 0 \Rightarrow k_1 = -1, k_2 = 1 \Rightarrow y_h = C_1 \cdot e^{-\beta} + C_2 \cdot e^{\beta}$

$$\Rightarrow \begin{cases} C_1^{'}(\beta) \cdot e^{-\beta} + C_2^{'}(\beta) \cdot e^{\beta} = 0, \\ -C_1^{'}(\beta) \cdot e^{-\beta} + C_2^{'}(\beta) \cdot e^{\beta} = \mathrm{th}\ \beta \end{cases}$$

$$\Rightarrow \begin{cases} C_1(\beta) = \dfrac{1}{2} \cdot \displaystyle\int \dfrac{(1 - e^{2\cdot\beta}) \cdot e^{\beta}}{e^{2\cdot\beta} + 1} \cdot \mathrm{d}\beta = \mathrm{arctg}(e^{\beta}) - \dfrac{1}{2} \cdot e^{\beta} + C_1, \\ C_2(\beta) = \dfrac{1}{2} \cdot \displaystyle\int \dfrac{1 - e^{2\cdot\beta}}{e^{\beta} \cdot (e^{2\cdot\beta} + 1)} \cdot \mathrm{d}\beta = \int \dfrac{e^{\beta}}{e^{2\cdot\beta} + 1} \cdot \mathrm{d}\beta - \dfrac{1}{2} \cdot \int e^{-\beta} \cdot \mathrm{d}\beta = \mathrm{arctg}(e^{\beta}) + \dfrac{1}{2} \cdot e^{-\beta} + C_2 \end{cases}$$

$$\Rightarrow y = C_1 \cdot e^{-\beta} + C_2 \cdot e^{\beta} + (e^{\beta} + e^{-\beta}) \cdot \mathrm{arctg}(e^{\beta}) = C_1 \cdot e^{-\beta} + C_2 \cdot e^{\beta} + 2 \cdot \mathrm{ch}\ \beta \cdot \mathrm{arctg}(e^{\beta}).$$