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1. Ispišite prvih pet članova svakoga od sljedećih nizova:

a)  $a_n = n^2 + 1$ ;

b)  $b_n = \sin\left(n \cdot \frac{\pi}{2}\right)$ ;


c)  $c_n = \frac{1}{n}$ .

*Rješenje:* U pravilo svakoga niza uvrštavamo  $n \in \{1, 2, 3, 4, 5\}$ . Dobivamo:

a)  $a_1 = 1^2 + 1 = 2$ ,  $a_2 = 2^2 + 1 = 5$ ,  $a_3 = 3^2 + 1 = 10$ ,  $a_4 = 4^2 + 1 = 17$ ,  $a_5 = 5^2 + 1 = 26$ .


b)  $b_1 = \sin\left(\frac{\pi}{2}\right) = 1$ ,  $b_2 = \sin \pi = 0$ ,  $b_3 = \sin\left(\frac{3}{2} \cdot \pi\right) = -1$ ,  $b_4 = \sin(2 \cdot \pi) = 0$ ,  $b_5 = \sin\left(\frac{5}{2} \cdot \pi\right) = 1$ .

c)  $c_1 = \frac{1}{1} = 1$ ,  $c_2 = \frac{1}{2}$ ,  $c_3 = \frac{1}{3}$ ,  $c_4 = \frac{1}{4}$ ,  $c_5 = \frac{1}{5}$ .

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**2. Isključivo pomoću definicije limesa niza dokažite da je  $\lim_n \left(\frac{1}{n}\right) = 0$ .**

*Rješenje:* Neka je  $\varepsilon > 0$  proizvoljan, ali fiksiran. Tada tražimo  $n_0 = n_0(\varepsilon)$  takav da za svaki  $n \in \mathbb{N}$  koji je veći od  $n_0$  vrijedi nejednakost  $\left|\frac{1}{n} - 0\right| < \varepsilon$ . Ta nejednakost je ekvivalentna nejednakosti  $\left|\frac{1}{n}\right| < \varepsilon$ , odnosno  $\frac{1}{n} < \varepsilon$ . Odatle je  $n > \frac{1}{\varepsilon}$ , pa vidimo da možemo uzeti  $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$ . Time je dokazana tvrdnja  $\lim_n \left(\frac{1}{n}\right) = 0$ .

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3. Zadani su nizovi  $a_n = \frac{4 \cdot n + 1}{1 - 2 \cdot n}$  i  $b_n = \frac{1 - 4 \cdot n}{12 \cdot n + 5}$ . Neka su  $L_1$  i  $L_2$  redom njihovi limesi.

a) Odredite  $L_1$  i  $L_2$ .

b) Odredite najmanji  $k \in \mathbb{N}$  takav da je  $|a_k - L_1| < 0.0001$ .

c) Odredite najmanji  $l \in \mathbb{N}$  takav da je  $|b_l - L_2| < 0.0001$ .

*Rješenje:* a) Podijelimo brojnik i nazivnik s  $n$ , pa primijenimo rezultat prethodnoga zadatka. Dobivamo:

$$L_1 = \lim_n \left( \frac{4 \cdot n + 1}{1 - 2 \cdot n} \right) = \lim_n \left( \frac{4 + \frac{1}{n}}{\frac{1}{n} - 2} \right) = \frac{\lim_n (4) + \lim_n \left( \frac{1}{n} \right)}{\lim_n \left( \frac{1}{n} \right) - \lim_n (2)} = \frac{4 + 0}{0 - 2} = -2,$$


$$L_2 = \lim_n \left( \frac{1 - 4 \cdot n}{12 \cdot n + 5} \right) = \lim_n \left( \frac{\frac{1}{n} - 4}{12 + 5 \cdot \frac{1}{n}} \right) = \frac{\lim_n \left( \frac{1}{n} \right) - \lim_n (4)}{\lim_n (12) + \lim_n \left( \frac{5}{n} \right)} = \frac{0 - 4}{12 + 0} = -\frac{1}{3}.$$

b) U nejednakost  $|a_k - L_1| < 0.0001$  uvrstimo pravilo niza  $a_n$  (pri čemu umjesto varijable  $n$  pišemo varijablu  $k$ ) i  $L_1 = -2$ . Dobivamo:

$$\begin{aligned} \left| \frac{4 \cdot k + 1}{1 - 2 \cdot k} + 2 \right| < 0.0001 &\Leftrightarrow \left| \frac{3}{1 - 2 \cdot k} \right| < 0.0001 \Leftrightarrow \frac{|3|}{\underbrace{|1 - 2 \cdot k|}_{<0}} < 0.0001 \Leftrightarrow \frac{3}{2 \cdot k - 1} < 0.0001 \\ &\Leftrightarrow 2 \cdot k - 1 > 30\,000 \Leftrightarrow 2 \cdot k > 30\,001 \Leftrightarrow k > 15\,000.5 \Rightarrow k_{\min} = 15\,001. \end{aligned}$$

c) Postupimo analogno kao u prethodnom podzadatku. Dobivamo:

$$\begin{aligned} \left| \frac{1 - 4 \cdot l}{12 \cdot l + 5} + \frac{1}{3} \right| < 0.0001 &\Leftrightarrow \left| \frac{(1 - 4 \cdot l) \cdot 3 + 1 \cdot (12 \cdot l + 5)}{3 \cdot (12 \cdot l + 5)} \right| < 0.0001 \Leftrightarrow \left| \frac{8}{3 \cdot (12 \cdot l + 5)} \right| < 0.0001 \Leftrightarrow \\ &\Leftrightarrow \frac{|8|}{\underbrace{3 \cdot (12 \cdot l + 5)}_{>0}} < 0.0001 \Leftrightarrow \frac{8}{36 \cdot l + 15} < 0.0001 \Leftrightarrow 36 \cdot l + 15 > 80\,000 \Leftrightarrow l > 2\,221.8 \Rightarrow l_{\min} = 2\,222. \end{aligned}$$

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4. Odredite limese sljedećih nizova:

a)  $a_n = \frac{2 \cdot n^2 + 3 \cdot n + 1}{4 \cdot n^2 - 5 \cdot n - 1};$

*Rješenje:* Podijelimo brojnik i nazivnik pravila niza s  $n^2$ , pa primijenimo svojstvo  $\lim_n \left( \frac{1}{n} \right) = \lim_n \left( \frac{1}{n^2} \right) = 0$ . Dobivamo:

$$L = \lim_n a_n = \lim_n \left( \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{4 - \frac{5}{n} - \frac{1}{n^2}} \right) = \frac{\lim_n (2) + \lim_n \left( \frac{3}{n} \right) + \lim_n \left( \frac{1}{n^2} \right)}{\lim_n (4) - \lim_n \left( \frac{5}{n} \right) - \lim_n \left( \frac{1}{n^2} \right)} = \frac{2 + 0 + 0}{4 - 0 - 0} = \frac{1}{2}.$$

b)  $b_n = \frac{2^{n+1} + 1}{2^{n-1} - 1};$

*Rješenje:* Podijelimo brojnik i nazivnik pravila niza s  $2^{n+1}$ , pa primijenimo jednakost  $\lim_n (a^n) = +\infty, \forall a > 1$ . Dobivamo:

$$L = \lim_n \left( \frac{1 + \frac{1}{2^{n+1}}}{\frac{2^{n-1}}{2^{n+1}} - \frac{1}{2^{n+1}}} \right) = \lim_n \left( \frac{1 + \frac{1}{2^{n+1}}}{2^{-2} - \frac{1}{2^{n+1}}} \right) = \frac{\lim_n (1) + \lim_n \left( \frac{1}{2^{n+1}} \right)}{\lim_n (2^{-2}) - \lim_n \left( \frac{1}{2^{n+1}} \right)} = \frac{1 + 0}{2^{-2} - 0} = \frac{1}{2^{-2}} = 2^2 = 4.$$


c)  $c_n = 2 \cdot (\sqrt{n + \sqrt{n}} - \sqrt{n});$

*Rješenje:* Pravilo niza najprije transformirajmo ovako:

$$(\sqrt{n + \sqrt{n}} - \sqrt{n}) \cdot \frac{(\sqrt{n + \sqrt{n}} + \sqrt{n})}{(\sqrt{n + \sqrt{n}} + \sqrt{n})} = \frac{(\sqrt{n + \sqrt{n}})^2 - (\sqrt{n})^2}{\sqrt{n + \sqrt{n}} + \sqrt{n}} = \frac{n + \sqrt{n} - n}{\sqrt{n + \sqrt{n}} + \sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n + \sqrt{n}} + \sqrt{n}}.$$

Sada podijelimo brojnik i nazivnik dobivenoga izraza s  $\sqrt{n}$ , pa primijenimo rezultat 2. zadatka. Dobivamo:

$$L = 2 \cdot \lim_n \left( \frac{1}{\sqrt{\frac{n}{n}} + \frac{\sqrt{n}}{n} + 1} \right) = 2 \cdot \lim_n \left( \frac{1}{\sqrt{1 + \sqrt{\frac{n}{n^2}}} + 1} \right) = 2 \cdot \lim_n \left( \frac{1}{\sqrt{1 + \sqrt{\frac{1}{n}}} + 1} \right) = 2 \cdot \frac{1}{\sqrt{1 + 0} + 1} = 2 \cdot \frac{1}{1 + 1} = 1.$$

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$$\text{d)} d_n = \frac{\sqrt{9 \cdot n^2 + 79 \cdot n} - \sqrt{73 \cdot n}}{n + \sqrt{4 \cdot n^2 + 1}};$$

*Rješenje:* Podijelimo brojnik i nazivnik pravila niza s  $n$ . Dobivamo:

$$\begin{aligned} L &= \lim_n \left( \frac{\sqrt{\frac{9 \cdot n^2 + 79 \cdot n}{n^2}} - \sqrt{\frac{73 \cdot n}{n^2}}}{\frac{n}{n} + \sqrt{\frac{4 \cdot n^2 + 1}{n^2}}} \right) = \lim_n \left( \frac{\sqrt{9 + \frac{79}{n}} - \sqrt{\frac{73}{n}}}{1 + \sqrt{4 + \frac{1}{n^2}}} \right) = \frac{\lim_n \left( \sqrt{9 + \frac{79}{n}} \right) - \lim_n \left( \sqrt{\frac{73}{n}} \right)}{\lim_n (1) + \lim_n \left( \sqrt{4 + \frac{1}{n^2}} \right)} = \\ &= \frac{\sqrt{9+0} - \sqrt{0}}{1 + \sqrt{4+0}} = \frac{3-0}{1+2} = 1. \end{aligned}$$

$$\text{e)} e_n = \left( 1 + \frac{1}{4 \cdot n} \right)^{2 \cdot n};$$

*Rješenje:* Koristimo jednakost  $\lim_n \left( \left( 1 + \frac{a}{n} \right)^n \right) = e^a, \forall a \in \mathbb{R}$ . Imamo redom:

$$L = \lim_n \left( 1 + \frac{1}{4 \cdot n} \right)^{2 \cdot n} = \lim_n \left( \left( 1 + \frac{\frac{1}{4}}{n} \right)^n \right)^2 = \left( e^{\frac{1}{4}} \right)^2 = e^{\frac{1}{2}} = \sqrt{e}.$$

$$\text{f)} f_n = \left( \frac{n+3}{n+2} \right)^{2 \cdot n+1}.$$

*Rješenje:* Pravilo zadanoga niza transformirajmo ovako:

$$\left( \frac{n+3}{n+2} \right)^{2n+1} = \left( \frac{(n+2)+1}{n+2} \right)^{2n+1} = \left( \frac{n+2}{n+2} + \frac{1}{n+2} \right)^{2n+1} = \left( 1 + \frac{1}{n+2} \right)^{2n+1} = \left( \left( 1 + \frac{1}{n+2} \right)^{n+2} \right)^{\frac{2n+1}{n+2}}.$$

Tako dobivamo:

$$L = \lim_n \left( \left( 1 + \frac{1}{n+2} \right)^{n+2} \right)^{\frac{2 \cdot n+1}{n+2}} = \left( \lim_n \left( \left( 1 + \frac{1}{n+2} \right)^{n+2} \right) \right)^{\lim_n \left( \frac{2 \cdot n+1}{n+2} \right)} = e^{\lim_n \left( \frac{2 + \frac{1}{n}}{1 + \frac{2}{n}} \right)} = e^{\frac{2+0}{1+0}} = e^2.$$

5. Odredite sljedeće limese:

a)  $\lim_{x \rightarrow 1} \left( \frac{x^2 - 1}{2 \cdot x - 2} \right);$

*Rješenje:* Rastavimo na faktore brojnik i nazivnik racionalne funkcije pod limesom. Imamo redom:

$$L = \lim_{x \rightarrow 1} \left( \frac{(x-1) \cdot (x+1)}{2 \cdot (x-1)} \right) = \lim_{x \rightarrow 1} \left( \frac{x+1}{2} \right) = \frac{1+1}{2} = 1.$$

b)  $\lim_{x \rightarrow (-2)} \left( \frac{x^3 + 8}{4 \cdot x + 8} \right);$

*Rješenje:* Postupimo analogno kao u prethodnom podzadatku. Dobivamo:

$$L = \lim_{x \rightarrow -2} \left( \frac{(x+2) \cdot (x^2 - 2 \cdot x + 4)}{4 \cdot (x+2)} \right) = \lim_{x \rightarrow -2} \left( \frac{x^2 - 2 \cdot x + 4}{4} \right) = \frac{(-2)^2 - 2 \cdot (-2) + 4}{4} = 3.$$

c)  $\lim_{x \rightarrow 4} \left( \frac{4 \cdot \sqrt{x} - 8}{x - 4} \right);$

*Rješenje:* Primijetimo da vrijedi jednakost  $x - 4 = (\sqrt{x})^2 - 2^2 = (\sqrt{x} - 2) \cdot (\sqrt{x} + 2)$ , za svaki  $x \geq 0$ . Tako sada imamo:

$$L = \lim_{x \rightarrow 4} \frac{4 \cdot (\sqrt{x} - 2)}{(\sqrt{x} - 2) \cdot (\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \left( \frac{4}{\sqrt{x} + 2} \right) = \frac{4}{\sqrt{4} + 2} = 1.$$

d)  $\lim_{x \rightarrow 0} \left( \frac{4 \cdot \sin x}{\sin(2 \cdot x)} \right).$

*Rješenje:* Primjenom identiteta  $\sin(2 \cdot x) = 2 \cdot \sin x \cdot \cos x$ ,  $\forall x \in \mathbb{R}$ , dobivamo:

$$L = 4 \cdot \lim_{x \rightarrow 0} \left( \frac{\sin x}{2 \cdot \sin x \cdot \cos x} \right) = \frac{4}{2} \cdot \lim_{x \rightarrow 0} \left( \frac{1}{\cos x} \right) = 2 \cdot \frac{1}{\cos 0} = 2 \cdot \frac{1}{1} = 2.$$

6. Odredite sljedeće limese:

a)  $\lim_{x \rightarrow \frac{\pi}{2}} \left( \frac{\cos x}{\pi - 2 \cdot x} \right);$

*Rješenje:* Uvedimo zamjenu  $t := x - \frac{\pi}{2}$ . Kad  $x \rightarrow \frac{\pi}{2}$ , onda  $t \rightarrow 0$ . Nadalje, iz  $t := x - \frac{\pi}{2}$  slijedi  $x = t + \frac{\pi}{2}$ . Tako imamo:

$$L = \lim_{t \rightarrow 0} \left( \frac{\cos \left( t + \frac{\pi}{2} \right)}{\pi - 2 \cdot \left( t + \frac{\pi}{2} \right)} \right) = \lim_{t \rightarrow 0} \left( \frac{\cos t \cdot \cos \frac{\pi}{2} - \sin t \cdot \sin \frac{\pi}{2}}{\pi - 2 \cdot t - \pi} \right) = \lim_{t \rightarrow 0} \left( \frac{-\sin t}{-2 \cdot t} \right) = \lim_{t \rightarrow 0} \left( \frac{\sin t}{2 \cdot t} \right) = \frac{1}{2}.$$

b)  $\lim_{x \rightarrow 0} \left( \frac{e^{3 \cdot x} - e^x}{2 \cdot x} \right);$

*Rješenje:* Koristimo identitet:

$$e^{3 \cdot x} - e^x = e^x \cdot (e^{2 \cdot x} - 1) = e^x \cdot ((e^x)^2 - 1^2) = e^x \cdot (e^x - 1) \cdot (e^x + 1), \quad \forall x \in \mathbb{R}.$$

Tako dobivamo:


$$L = \lim_{x \rightarrow 0} \frac{e^x \cdot (e^x - 1) \cdot (e^x + 1)}{2 \cdot x} = \left( \lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) \right) \cdot \left( \lim_{x \rightarrow 0} \left( \frac{e^x \cdot (e^x + 1)}{2} \right) \right) = 1 \cdot \frac{1 \cdot (1 + 1)}{2} = 1.$$

c)  $\lim_{x \rightarrow +\infty} \left( x - \sqrt{x^2 - 4 \cdot x + 5} \right);$

*Rješenje:* Funkciju pod limesom transformiramo ovako:

$$\begin{aligned} & \left( x - \sqrt{x^2 - 4 \cdot x + 5} \right) \cdot \frac{\left( x + \sqrt{x^2 - 4 \cdot x + 5} \right)}{\left( x + \sqrt{x^2 - 4 \cdot x + 5} \right)} = \frac{x^2 - \left( \sqrt{x^2 - 4 \cdot x + 5} \right)^2}{x + \sqrt{x^2 - 4 \cdot x + 5}} = \frac{x^2 - (x^2 - 4 \cdot x + 5)}{x + \sqrt{x^2 - 4 \cdot x + 5}} = \\ & = \frac{x^2 - x^2 + 4 \cdot x - 5}{x + \sqrt{x^2 - 4 \cdot x + 5}} = \frac{4 \cdot x - 5}{x + \sqrt{x^2 - 4 \cdot x + 5}}. \end{aligned}$$

Brojnik i nazivnik dobivenoga izraza podijelimo s  $x$ , pa dobijemo:

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$$\frac{4 \cdot x - 5}{x + \sqrt{x^2 - 4 \cdot x + 5}} = \frac{4 - 5 \cdot \frac{1}{x}}{1 + \sqrt{\frac{x^2 - 4 \cdot x + 5}{x^2}}} = \frac{4 - 5 \cdot \frac{1}{x}}{1 + \sqrt{1 - 4 \cdot \frac{1}{x} + 5 \cdot \frac{1}{x^2}}}.$$

Zbog toga je traženi limes jednak:

$$L = \lim_{x \rightarrow +\infty} \left( \frac{4 - 5 \cdot \frac{1}{x}}{1 + \sqrt{1 - 4 \cdot \frac{1}{x} + 5 \cdot \frac{1}{x^2}}} \right) = \frac{4 - 0}{1 + \sqrt{1 - 0 + 0}} = \frac{4}{1 + 1} = 2.$$

**d)**  $\lim_{x \rightarrow -\infty} \left( x^2 \cdot (\ln(x^2 - 1) - \ln(x^2 + 1)) \right);$

*Rješenje:* Funkciju pod limesom transformiramo ovako:

$$x^2 \cdot (\ln(x^2 - 1) - \ln(x^2 + 1)) = x^2 \cdot \ln \left( \frac{x^2 - 1}{x^2 + 1} \right) = \ln \left( \left( \frac{x^2 - 1}{x^2 + 1} \right)^{x^2} \right).$$

Uvedimo zamjenu  $t := x^2 + 1$ . Kad  $x \rightarrow -\infty$ , onda  $t \rightarrow +\infty$ . Nadalje, iz  $t := x^2 + 1$  slijedi  $x^2 = t - 1$ . Tako je traženi limes jednak:

$$\begin{aligned} L &= \ln \left( \lim_{t \rightarrow +\infty} \left( \left( \frac{(t-1)-1}{t} \right)^{t-1} \right) \right) = \ln \left( \lim_{t \rightarrow +\infty} \left( \left( \frac{t-2}{t} \right)^{t-1} \right) \right) = \ln \left( \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^t \cdot \left( 1 - \frac{2}{t} \right)^{-1} \right) \right) = \\ &= \ln \left( \left( \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^t \right) \right) \cdot \left( \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^{-1} \right) \right) \right) = \ln(e^{-2} \cdot (1-0)^{-1}) = \ln(e^{-2}) = -2. \end{aligned}$$

**e)**  $\lim_{x \rightarrow +\infty} \left( \left( \frac{x}{x+2} \right)^{6 \cdot x} \right).$

*Rješenje:* Postupimo analogno kao u rješenju prethodnoga podzadatka:

$$\begin{aligned} L &= \lim_{x \rightarrow +\infty} \left( \left( \frac{(x+2)-2}{x+2} \right)^{6 \cdot x} \right) = \lim_{x \rightarrow +\infty} \left( \left( \frac{x+2}{x+2} - \frac{2}{x+2} \right)^{6 \cdot x} \right) = \lim_{x \rightarrow +\infty} \left( \left( 1 - \frac{2}{x+2} \right)^{6 \cdot x} \right) = \left\{ \begin{array}{l} \text{zamjena: } t := x+2, \\ \text{kad } x \rightarrow +\infty, \text{ onda } t \rightarrow +\infty, \\ x = t-2 \end{array} \right\} = \\ &= \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^{6(t-2)} \right) = \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^{6t-12} \right) = \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^{6t} \cdot \left( 1 - \frac{2}{t} \right)^{-12} \right) = \lim_{t \rightarrow +\infty} \left( \left( \left( 1 - \frac{2}{t} \right)^t \right)^6 \cdot \left( 1 - \frac{2}{t} \right)^{-12} \right) = \\ &= \left( \lim_{t \rightarrow +\infty} \left( \left( \left( 1 - \frac{2}{t} \right)^t \right)^6 \right) \right) \cdot \left( \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^{-12} \right) \right) = \left( \lim_{t \rightarrow +\infty} \left( \left( 1 - \frac{2}{t} \right)^t \right) \right)^6 \cdot \left( \lim_{t \rightarrow +\infty} \left( 1 - \frac{2}{t} \right) \right)^{-12} = (e^{-2})^6 \cdot (1-0)^{-12} = e^{-12} \cdot 1 = e^{-12}. \end{aligned}$$



7. Odredite sljedeće jednostrane limese:

a)  $\lim_{x \rightarrow 2-} \left( \arctg \left( \frac{1}{x-2} \right) \right);$

*Rješenje:* Kad  $x \rightarrow 2-$ , onda  $x-2 \rightarrow 0-$ , pa  $\frac{1}{x-2} \rightarrow -\infty$ , te  $\arctg \left( \frac{1}{x-2} \right) \rightarrow -\frac{\pi}{2}$ .

Dakle, traženi je limes jednak  $-\frac{\pi}{2}$ .

b)  $\lim_{x \rightarrow 5+} \left( \arctg \left( \frac{1}{x-5} \right) \right);$

*Rješenje:* Kad  $x \rightarrow 5+$ , onda  $x-5 \rightarrow 0+$ , pa  $\frac{1}{x-5} \rightarrow +\infty$ , te  $\arctg \left( \frac{1}{x-5} \right) \rightarrow \frac{\pi}{2}$ .

Dakle, traženi je limes jednak  $\frac{\pi}{2}$ .

c)  $\lim_{x \rightarrow 3+} \left( 2^{\frac{1}{3-x}} \right);$

*Rješenje:* Kad  $x \rightarrow 3+$ , onda  $3-x \rightarrow 0-$ , pa  $\frac{1}{3-x} \rightarrow -\infty$ , te  $2^{\frac{1}{3-x}} \rightarrow 0$ . Dakle, traženi je limes jednak 0.

d)  $\lim_{x \rightarrow 4+} \left( 5^{\frac{1}{\sqrt{x}-2}} \right).$

*Rješenje:* Kad  $x \rightarrow 4+$ , onda  $\sqrt{x}-2 \rightarrow 0+$ , pa  $\frac{1}{\sqrt{x}-2} \rightarrow +\infty$ , te  $5^{\frac{1}{\sqrt{x}-2}} \rightarrow +\infty$ . Zbog toga ovaj limes ne postoji.

8. Navedite konkretan primjer realne funkcije  $f$  i točke  $c \in \mathbb{R}$  takvih da:

a) postoje  $\lim_{x \rightarrow c-} f(x)$  i  $\lim_{x \rightarrow c+} f(x)$ , a ne postoji  $f(c)$ ;

*Rješenje:* Npr. bilo koja funkcija iz zadatka 5. i točka u kojoj je računan dotični limes. Primijetite da u svakom od tih slučajeva vrijedi  $\lim_{x \rightarrow c-} f(x) = \lim_{x \rightarrow c+} f(x)$ .

b) postoje  $f(c)$  i  $\lim_{x \rightarrow c-} f(x)$ , a ne postoji  $\lim_{x \rightarrow c+} f(x)$ ;

*Rješenje:* Npr.  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \begin{cases} \sin x, & \text{za } x < 0, \\ 0, & \text{za } x = 0, \\ \frac{1}{x}, & \text{inače.} \end{cases}$  i  $c = 0$ . Očito su  $f(0) = 0$ ,  $\lim_{x \rightarrow 0-} f(x) = 0$ , ali  $\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} \left( \frac{1}{x} \right) = +\infty$ .

c) postoje  $f(c)$  i  $\lim_{x \rightarrow c+} f(x)$ , a ne postoji  $\lim_{x \rightarrow c-} f(x)$ .

*Rješenje:* Npr.  $g: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = \begin{cases} \frac{1}{x}, & \text{za } x < 0, \\ 0, & \text{za } x = 0, \\ \sin x, & \text{za } x > 0 \end{cases}$  i  $c = 0$ . Očito su  $g(0) = 0$ ,  $\lim_{x \rightarrow 0+} g(x) = 0$ , ali  $\lim_{x \rightarrow 0-} g(x) = \lim_{x \rightarrow 0-} \left( \frac{1}{x} \right) = -\infty$ .

9. Navedite konkretan primjer realnih funkcija  $f$  i  $g$ , te točke  $c \in \mathbb{R}$  takvih da ne postoje  $\lim_{x \rightarrow c} f(x)$  i  $\lim_{x \rightarrow c} g(x)$ , a postoji:

a)  $\lim_{x \rightarrow c} ((f + g)(x))$ ;

*Rješenje:* Npr.  $f(x) = 1 - \frac{1}{x}$ ,  $g(x) = \frac{1}{x}$  i  $c = 0$ . Očito su  $\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} \left(1 - \frac{1}{x}\right) = -\infty$ ,

$$\lim_{x \rightarrow 0-} f(x) = \lim_{x \rightarrow 0-} \left(1 - \frac{1}{x}\right) = +\infty, \quad \lim_{x \rightarrow 0+} g(x) = \lim_{x \rightarrow 0+} \left(\frac{1}{x}\right) = +\infty, \quad \lim_{x \rightarrow 0-} g(x) = \lim_{x \rightarrow 0-} \left(\frac{1}{x}\right) = -\infty, \quad \text{ali}$$

$$\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} \left(1 - \frac{1}{x} + \frac{1}{x}\right) = \lim_{x \rightarrow 0} (1) = 1.$$

b)  $\lim_{x \rightarrow c} ((f \cdot g)(x))$ .

*Rješenje:* Npr.  $f(x) = \begin{cases} \sin x, & \text{za } x < 0, \\ 1, & \text{za } x = 0, \\ \frac{1}{x}, & \text{za } x > 0, \end{cases}$  ,  $g(x) = \begin{cases} \frac{1}{x}, & \text{za } x < 0, \\ 1, & \text{za } x = 0, \\ \sin x, & \text{za } x > 0, \end{cases}$  i  $c = 0$ . U zadacima

8. b) i c) pokazali smo da ne postoje  $\lim_{x \rightarrow c} f(x)$  i  $\lim_{x \rightarrow c} g(x)$ . Međutim,

$$(f \cdot g)(x) = \begin{cases} \frac{\sin x}{x}, & \text{za } x \neq 0, \\ 1, & \text{za } x = 0, \end{cases}$$

pa je

$$\lim_{x \rightarrow 0} ((f \cdot g)(x)) = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right) = \frac{1}{1} = 1 = (f \cdot g)(0).$$