

1. Ispišite prvih pet članova svakoga od sljedećih nizova:

a) $a_n = n^2 + 1$;

b) $b_n = \sin\left(n \cdot \frac{\pi}{2}\right)$;

c) $c_n = \frac{1}{n}$.

2. Izračunajte granične vrijednosti sljedećih nizova:

a) $a_n = \frac{2 \cdot n + 1}{n - 1}$;

b) $b_n = \frac{2 \cdot n^2 + n + 1}{4 \cdot n^2 - n - 1}$;

c) $c_n = \frac{\sqrt{n^2 + n} - \sqrt{n}}{n + \sqrt{n^2 + 1}}$;

d) $d_n = \frac{\sqrt[3]{n^2 - 1} + \sqrt[3]{n}}{\sqrt[3]{8 \cdot n^2 + n} - 4 \cdot \sqrt[3]{n}}$;

e) $e_n = \left(1 + \frac{1}{n}\right)^{2n}$;

f) $f_n = \left(1 + \frac{1}{4 \cdot n}\right)^{16n}$.

3. Zadani su nizovi $a_n = \frac{2 \cdot n + 1}{1 - n}$ i $b_n = \frac{1 - 4 \cdot n}{12 \cdot n + 5}$. Neka su L_1 i L_2 redom njihove granične vrijednosti.

a) Izračunajte L_1 i L_2 .

b) Odredite najmanji $k \in \mathbb{N}$ takav da je $|a_k - L_1| < 0.0001$.

c) Odredite najmanji $l \in \mathbb{N}$ takav da je $|b_l - L_2| < 0.0001$.

4. Izračunajte granične vrijednosti:

a) $\lim_{x \rightarrow 0} \frac{x^2 + x}{x}$;

b) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$;

c) $\lim_{x \rightarrow -1} \frac{x^3 + 1}{2 \cdot x + 2}$;

d) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$.

5. Izračunajte sljedeće granične vrijednosti:

a) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2 \cdot x}$;

b) $\lim_{x \rightarrow 1} (1 - x) \cdot \operatorname{tg} \left(\frac{\pi}{2} \cdot x \right)$;

c) $\lim_{x \rightarrow 0} \left(\frac{x^2 - 2 \cdot x + 4}{x^2 - 3 \cdot x + 2} \right)^{\frac{x}{\sin x}}$;

d) $\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}}$;

e) $\lim_{x \rightarrow +\infty} \{x \cdot [\ln(x+1) - \ln x]\}$

6. Izračunajte sljedeće jednostrane granične vrijednosti:

a) $\lim_{x \rightarrow +\infty} \frac{\sin x}{x}$;

b) $\lim_{x \rightarrow -\infty} \frac{x \cdot \cos x}{x^2 + 1}$;

c) $\lim_{x \rightarrow +\infty} \frac{x^2 \cdot \operatorname{arctg} x}{x^2 + x + 1}$;

d) $\lim_{x \rightarrow -\infty} \frac{x^3 \cdot \operatorname{arcctg} x}{x^3 + x^2 + x + 1}$.

7. Isključivo pomoću definicije granične vrijednosti niza dokažite da je $\lim_n \frac{1}{n} = 0$. Potom

utvrdite što se dobiva ako bismo uzeli da je $\lim_n \frac{1}{n} = 10^{-10}$.

Rezultati zadataka

1.

a) $a_1 = 1^2 + 1 = 2$, $a_2 = 2^2 + 1 = 5$, $a_3 = 3^2 + 1 = 10$, $a_4 = 4^2 + 1 = 17$, $a_5 = 5^2 + 1 = 26$.

b) $b_1 = \sin\left(\frac{\pi}{2}\right) = 1$, $b_2 = \sin \pi = 0$, $b_3 = \sin\left(\frac{3}{2} \cdot \pi\right) = -1$, $b_4 = \sin(2 \cdot \pi) = 1$, $b_5 = \sin\left(\frac{5}{2} \cdot \pi\right) = 1$.

c) $c_1 = \frac{1}{1} = 1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{3}$, $c_4 = \frac{1}{4}$, $c_5 = \frac{1}{5}$.

2.

a)
$$\lim_n a_n = \lim_n \frac{2 \cdot n + 1}{n - 1} = \lim_n \frac{2 + \frac{1}{n}}{1 - \frac{1}{n}} = \frac{\lim_n(2) + \lim_n\left(\frac{1}{n}\right)}{\lim_n(1) - \lim_n\left(\frac{1}{n}\right)} = \frac{2 + 0}{1 - 0} = 2;$$

b)
$$\lim_n b_n = \lim_n \frac{2 \cdot n^2 + n + 1}{4 \cdot n^2 - n - 1} = \lim_n \frac{2 + \frac{1}{n} + \frac{1}{n^2}}{4 - \frac{1}{n} - \frac{1}{n^2}} = \frac{\lim_n(2) + \lim_n\left(\frac{1}{n}\right) + \lim_n\left(\frac{1}{n^2}\right)}{\lim_n(4) - \lim_n\left(\frac{1}{n}\right) - \lim_n\left(\frac{1}{n^2}\right)} = \frac{2 + 0 + 0}{4 - 0 - 0} = \frac{1}{2};$$

c)
$$\lim_n c_n = \lim_n \frac{\sqrt{n^2 + n} - \sqrt{n}}{n + \sqrt{n^2 + 1}} = \lim_n \frac{\sqrt{1 + \frac{1}{n}} - \sqrt{\frac{1}{n}}}{1 + \sqrt{1 + \frac{1}{n}}} = \frac{\lim_n\left(\sqrt{1 + \frac{1}{n}}\right) - \lim_n\left(\sqrt{\frac{1}{n}}\right)}{\lim_n 1 + \lim_n\left(\sqrt{1 + \frac{1}{n}}\right)} = \frac{1 - 0}{1 + 1} = \frac{1}{2};$$

d)
$$\lim_n d_n = \lim_n \frac{\sqrt[3]{n^2 + 1} + \sqrt[3]{n}}{\sqrt[3]{8 \cdot n^2 + 1} - 4 \cdot \sqrt[3]{n}} = \lim_n \frac{\sqrt[3]{1 + \frac{1}{n^2}} + \sqrt[3]{\frac{1}{n}}}{\sqrt[3]{8 + \frac{1}{n^2}} - 4 \cdot \sqrt[3]{\frac{1}{n}}} = \frac{\lim_n\left(\sqrt[3]{1 + \frac{1}{n^2}}\right) + \lim_n\left(\sqrt[3]{\frac{1}{n}}\right)}{\lim_n\left(\sqrt[3]{8 + \frac{1}{n^2}}\right) - \lim_n\left(4 \cdot \sqrt[3]{\frac{1}{n}}\right)} = \frac{1 + 0}{2 - 0} = \frac{1}{2};$$

e)
$$\lim_n e_n = \lim_n \left(1 + \frac{1}{n}\right)^{2n} = \lim_n \left[\left(1 + \frac{1}{n}\right)^n\right]^2 = \left[\lim_n \left(1 + \frac{1}{n}\right)^n\right]^2 = e^2;$$

f)
$$\lim_n f_n = \lim_n \left(1 + \frac{1}{4 \cdot n}\right)^{16n} = \lim_n \left[\left(1 + \frac{1}{4}\right)^n\right]^{16} = \left(e^4\right)^{16} = e^4.$$

3.

$$L_1 = \lim_n \left(\frac{4 \cdot n + 1}{1 - 2 \cdot n}\right) = \lim_n \frac{4 + \frac{1}{n}}{\frac{1}{n} - 2} = \frac{\lim_n(4) + \lim_n\left(\frac{1}{n}\right)}{\lim_n\left(\frac{1}{n}\right) - \lim_n(2)} = \frac{4 + 0}{0 - 2} = -2.$$

$$|a_k - L_1| < 0.0001 \Leftrightarrow \left|\frac{4 \cdot k + 1}{1 - 2 \cdot k} + 2\right| < 0.0001 \Leftrightarrow \left|\frac{3}{1 - 2 \cdot k}\right| < 0.0001 \Leftrightarrow \frac{3}{2 \cdot k - 1} < 0.0001 \Leftrightarrow 2 \cdot k - 1 > 30000 \Leftrightarrow k > 15000.5 \Rightarrow k_{\min} = 15001$$

$$L_2 = \lim_n \left(\frac{1-4 \cdot n}{12 \cdot n + 5} \right) = \lim_n \frac{\frac{1}{n} - 4}{12 + \frac{5}{n}} = \frac{\lim_n \left(\frac{1}{n} \right) - \lim_n (4)}{\lim_n (12) + \lim_n \left(\frac{5}{n} \right)} = \frac{0-4}{12+0} = -\frac{1}{3}$$

$$|b_l - L_2| < 0.0001 \Leftrightarrow \left| \frac{1-4 \cdot l}{12 \cdot l + 5} + \frac{1}{3} \right| < 0.0001 \Leftrightarrow \left| \frac{8}{3 \cdot (12 \cdot l + 5)} \right| < 0.0001 \Leftrightarrow \frac{8}{36 \cdot l + 15} < 0.0001 \Leftrightarrow$$

$$\Leftrightarrow 36 \cdot l + 15 > 80000 \Leftrightarrow l > 2221.8 \Rightarrow l_{\min} = 2222$$

4.

a) $\lim_{x \rightarrow 0} \frac{x^2 + x}{x} = \lim_n \frac{x_n^2 + x_n}{x_n} = \lim_n (x_n + 1) = \lim_n (x_n) + \lim_n (1) = 0 + 1 = 1;$

b) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_n \frac{x_n^2 - 1}{x_n - 1} = \lim_n \frac{(x_n - 1) \cdot (x_n + 1)}{x_n - 1} = \left[\lim_n \frac{x_n - 1}{x_n - 1} \right] \cdot \left[\lim_n (x_n + 1) \right] = \left[\lim_n (1) \right] \cdot \left[\lim_n (x_n) + \lim_n (1) \right] = 1 \cdot (1 + 1) = 2;$

c) $\lim_{x \rightarrow -1} \frac{x^3 + 1}{2 \cdot x + 2} = \lim_n \frac{x_n^3 + 1}{2 \cdot x_n + 2} = \lim_n \frac{(x_n + 1) \cdot (x_n^2 - x_n + 1)}{2 \cdot (x_n + 1)} = \lim_n \left[\frac{x_n + 1}{2 \cdot (x_n + 1)} \right] \cdot \left[\lim_n (x_n^2 - x_n + 1) \right] =$

$$= \left[\lim_n \left(\frac{1}{2} \right) \right] \cdot \left[\lim_n (x_n^2) - \lim_n (x_n) + \lim_n (1) \right] = \frac{1}{2} \cdot [(-1)^2 - (-1) + 1] = \frac{3}{2};$$

d) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_n \frac{\sqrt{x_n} - 2}{x_n - 4} = \lim_n \frac{\sqrt{x_n} - 2}{(\sqrt{x_n})^2 - 2^2} = \lim_n \frac{\sqrt{x_n} - 2}{(\sqrt{x_n} - 2) \cdot (\sqrt{x_n} + 2)} = \left\{ \lim_n \left[\frac{\sqrt{x_n} - 2}{(\sqrt{x_n} - 2)} \right] \right\} \cdot \left\{ \lim_n \left[\frac{1}{\sqrt{x_n} + 2} \right] \right\} =$

$$= \left[\lim_n (1) \right] \cdot \left[\frac{\lim_n (1)}{\lim_n (\sqrt{x_n}) + \lim_n (2)} \right] = 1 \cdot \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.$$

5.

a) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2 \cdot x} = \left\{ t = x - \frac{\pi}{2} \right\} = \lim_{t \rightarrow 0} \frac{\cos \left(t + \frac{\pi}{2} \right)}{\pi - 2 \cdot \left(t + \frac{\pi}{2} \right)} = \lim_{t \rightarrow 0} \frac{-\sin t}{-2 \cdot t} = \frac{1}{2} \cdot \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right) = \frac{1}{2};$

b) $\lim_{x \rightarrow 1} \left[(1-x) \cdot \operatorname{tg} \left(\frac{\pi}{2} \cdot x \right) \right] = \{t = 1-x\} = \lim_{t \rightarrow 0} t \cdot \frac{\sin \left[\frac{\pi}{2} \cdot (1-t) \right]}{\cos \left[\frac{\pi}{2} \cdot (1-t) \right]} = \lim_{t \rightarrow 0} \left[t \cdot \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2} \cdot t \right)}{\cos \left(\frac{\pi}{2} - \frac{\pi}{2} \cdot t \right)} \right] =$

$$= \lim_{t \rightarrow 0} \left[t \cdot \frac{\cos \left(\frac{\pi}{2} \cdot t \right)}{\sin \left(\frac{\pi}{2} \cdot t \right)} \right] = \left[\lim_{t \rightarrow 0} \frac{t}{\sin \left(\frac{\pi}{2} \cdot t \right)} \right] \cdot \lim_{t \rightarrow 0} \left[\cos \left(\frac{\pi}{2} \cdot t \right) \right] = \frac{2}{\pi} \cdot 1 = \frac{2}{\pi};$$

c) $\lim_{x \rightarrow 0} \left(\frac{x^2 - 2 \cdot x + 4}{x^2 - 3 \cdot x + 2} \right)^{\frac{x}{\sin x}} = \left[\lim_{x \rightarrow 0} \left(\frac{x^2 - 2 \cdot x + 4}{x^2 - 3 \cdot x + 2} \right) \right]^{\left[\lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \right]} = \left(\frac{4}{2} \right)^1 = 2;$

d) $\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} [(1 + \sin x) - 1] \cdot \frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = e^1 = e;$

e) $\lim_{x \rightarrow +\infty} \{x \cdot [\ln(x+1) - \ln x]\} = \lim_{x \rightarrow +\infty} \ln \left[\left(\frac{x+1}{x} \right)^x \right] = \ln \lim_{x \rightarrow +\infty} \left[\left(1 + \frac{1}{x} \right)^x \right] = \ln e = 1.$

6.

$$\text{a) } -1 \leq \sin x \leq 1 \Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \Rightarrow \lim_{x \rightarrow +\infty} \left(-\frac{1}{x} \right) \leq \lim_{x \rightarrow +\infty} \frac{\sin x}{x} \leq \lim_{x \rightarrow +\infty} \frac{1}{x} \Rightarrow 0 \leq \lim_{x \rightarrow +\infty} \frac{\sin x}{x} \leq 0 \Rightarrow \lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0;$$

$$\text{b) } -1 \leq \cos x \leq 1 \Rightarrow -\frac{x}{x^2+1} \leq \frac{x \cdot \cos x}{x^2+1} \leq \frac{x}{x^2+1} \Rightarrow \lim_{x \rightarrow -\infty} \left(-\frac{x}{x^2+1} \right) \leq \lim_{x \rightarrow -\infty} \left(\frac{x \cdot \cos x}{x^2+1} \right) \leq \lim_{x \rightarrow -\infty} \left(\frac{x}{x^2+1} \right)$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow -\infty} \left(\frac{x \cdot \cos x}{x^2+1} \right) \leq 0 \Rightarrow \lim_{x \rightarrow -\infty} \left(\frac{x \cdot \cos x}{x^2+1} \right) = 0;$$

$$\text{c) } \lim_{x \rightarrow +\infty} \frac{x^2 \cdot \arctg x}{x^2+1} = \left[\lim_{x \rightarrow +\infty} \frac{x^2}{x^2+1} \right] \cdot \left[\lim_{x \rightarrow +\infty} (\arctg x) \right] = 1 \cdot \frac{\pi}{2} = \frac{\pi}{2};$$

$$\text{d) } \lim_{x \rightarrow +\infty} \frac{x^3 \cdot \text{arctg} x}{x^3+x^2+x+1} = \left[\lim_{x \rightarrow +\infty} \frac{x^3}{x^3+x^2+x+1} \right] \cdot \left[\lim_{x \rightarrow +\infty} (\text{arctg} x) \right] = 1 \cdot 0 = 0.$$

7. Neka je $\varepsilon > 0$ proizvoljan, ali fiksiran. Tada tražimo $n_0 = n_0(\varepsilon)$ takav da za svaki $n \in \mathbb{N}$

koji je veći od n_0 vrijedi nejednakost $\left| \frac{1}{n} - 0 \right| < \varepsilon$. Ta nejednakost je ekvivalentna nejednakosti

$\left| \frac{1}{n} \right| < \varepsilon$, odnosno $\frac{1}{n} < \varepsilon$. Odatle je $n > \frac{1}{\varepsilon}$, pa vidimo da možemo uzeti $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Time je

dokazana tvrdnja $\lim_n \frac{1}{n} = 0$.

Pretpostavimo da je $\lim_n \frac{1}{n} = 10^{-10}$. Analognim postupkom se dobiva $\left| \frac{1}{n} - 10^{-10} \right| < \varepsilon$,

odnosno $\left| \frac{1 - 10^{-10} \cdot n}{n} \right| < \varepsilon$, odnosno $\frac{|1 - 10^{-10} \cdot n|}{n} < \varepsilon$. Odatle je $-n \cdot \varepsilon < 1 - 10^{-10} \cdot n < n \cdot \varepsilon$.

Rješavanjem nejednadžbi $-n \cdot \varepsilon < 1 - 10^{-10} \cdot n$ i $1 - 10^{-10} \cdot n < n \cdot \varepsilon$ dobijemo da one vrijede samo za konačno mnogo $n \in \mathbb{N}$, što je nemoguće (ako je $\lim_n \frac{1}{n} = 10^{-10}$, onda mora postojati beskonačno mnogo $n \in \mathbb{N}$ za koje su istinite navedene nejednadžbe). Pretpostavka je, dakle,

bila pogrešna, pa je $\lim_n \frac{1}{n} \neq 10^{-10}$.