1. Odredite derivacije sljedećih realnih funkcija i pojednostavnite dobiveni izraz što više možete:

a) $f_1(x) = (x^2 + 1)^{2018}$;

b) $f_2(t) = A \cdot \sin(\omega \cdot t + \varphi)$;

c) $f_3(u) = \sin^2(2 \cdot u)$;

d) $f_4(v) = \sqrt{\sin v} + \sqrt[3]{\cos^2 v}$;

e) $f_5(w) = \operatorname{arcctg}\left(\dfrac{1+w}{1-w}\right)$;

f) $f_6(y) = e^{1-y^2}$;

g) $f_7(\alpha) = \sqrt{\ln \alpha} + \ln(\sqrt{\alpha + 1} - 1)$;

h) $f_8(\beta) = (\beta^2 - \beta + 1)^3 + e^{\sin \beta} + \operatorname{tg}\sqrt{\beta}$;

i) $f_9(\gamma) = \ln(\ln \gamma) + \arcsin\sqrt{2 \cdot \gamma} + \operatorname{arctg}(\operatorname{ch} \gamma)$.

2. Odredite derivacije sljedećih implicitno zadanih funkcija i pojednostavnite dobiveni izraz što više možete:

a) $b^2 \cdot x^2 + a^2 \cdot y^2 - a^2 \cdot b^2 = 0$;

b) $\sqrt{x} + \sqrt[3]{\sin y} = a \cdot \sqrt{a}$;

c) $y^2 = \dfrac{x+y}{x-y}$;

d) $\operatorname{ctg} x = x \cdot y^2$;

e) $\ln x + e^{-\frac{y}{x}} = a$;

f) $\operatorname{arctg}\dfrac{y}{x} = \dfrac{1}{2} \cdot \ln(x^2 + y^2)$.

3. Pomoću logaritamskoga deriviranja odredite derivacije sljedećih realnih funkcija i pojednostavnite dobiveni izraz što više možete:

a) $f_1(x) = \dfrac{(x-2)^2}{(x-1)^4 \cdot (x-3)^3}$;

b) $f_2(t) = t \cdot \sqrt[4]{\dfrac{t^3}{t^2+1}}$;

c) $f_3(u) = u^u$;

d) $f_4(v) = v^{\sqrt[3]{v^2}}$

e) $f_5(w) = (\cos w)^{\sin w}$;

f) $f_6(y) = \left(1 + \dfrac{2}{y}\right)^y$.

4. Primjenom formule za derivaciju složene funkcije ili formule za derivaciju inverza funkcije dokažite sljedeće jednakosti:

a) $(\cos x)' = -\sin x$;

b) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$;

c) $(\arccos y)' = -\dfrac{1}{\sqrt{1-y^2}}$;

d) $(\operatorname{arctg} t)' = \dfrac{1}{1+t^2}$;

e) $(\operatorname{arcctg} u)' = -\dfrac{1}{1+u^2}$;

f) $(\ln v)' = \dfrac{1}{v}$.

5. Odredite derivacije sljedećih parametarski zadanih funkcija:

a) $\begin{cases} x = 4\cdot t - 1; \\ y = t^2; \end{cases}$

b) $\begin{cases} x = \dfrac{2\cdot t}{1+t^2}; \\ y = \dfrac{1-t^2}{1+t^2}; \end{cases}$

c) $\begin{cases} x = \sqrt{t}; \\ y = \sqrt[3]{t^2}; \end{cases}$

d) $\begin{cases} x = a\cdot \cos^3 t; \\ y = b\cdot \sin^3 t; \end{cases}$

e) $\begin{cases} x = e^{-t}; \\ y = e^{2\cdot t}; \end{cases}$

f) $\begin{cases} x = e^t \cdot \cos t; \\ y = e^t \cdot \sin t. \end{cases}$

Rezultati zadataka

1.

a) $f_1^{'}(x) = 4036 \cdot x \cdot (x^2+1)^{2017}$;

b) $f_2^{'}(t) = A \cdot \omega \cdot \cos(\omega \cdot t + \varphi)$;

c) $f_3^{'}(u) = 4 \cdot \sin(2 \cdot u) \cdot \cos(2 \cdot u) = 2 \cdot \sin(4 \cdot u)$;

d) $f_4^{'}(v) = \frac{\cos v}{2 \cdot \sqrt{\sin v}} - \frac{2 \cdot \sin v}{3 \cdot \sqrt[3]{\cos v}}$;

e) $f_5^{'}(w) = -\frac{\left(\frac{1+w}{1-w}\right)^{'}}{\left(\frac{1+w}{1-w}\right)^2 + 1} = -\frac{\frac{1 \cdot (1-w) - (1+w) \cdot (-1)}{(1-w)^2}}{\frac{(1+w)^2 + (1-w)^2}{(1-w)^2}} = -\frac{1}{w^2+1}$;

f) $f_6^{'}(y) = -2 \cdot y \cdot e^{1-y^2}$;

g) $f_7^{'}(\alpha) = \frac{1}{2 \cdot \alpha \cdot \sqrt{\ln \alpha}} + \frac{1}{2 \cdot \left(\sqrt{\alpha+1} - 1\right) \cdot \sqrt{\alpha+1}} = \frac{\sqrt{\ln \alpha}}{2 \cdot \alpha \cdot \ln \alpha} + \frac{\alpha + 1 + \sqrt{\alpha+1}}{2 \cdot \alpha \cdot (\alpha+1)}$;

h) $f_8^{'}(\beta) = 3 \cdot (2 \cdot \beta - 1) \cdot (\beta^2 - \beta + 1)^2 + \frac{\sqrt{\beta}}{2 \cdot \cos^2(\sqrt{\beta})} + \cos \beta \cdot e^{\sin \beta}$;

i) $f_9^{'}(\gamma) = \frac{1}{\gamma \cdot \ln \gamma} + \frac{1}{\sqrt{2 \cdot \gamma - 4 \cdot \gamma^2}} + \frac{\operatorname{sh} \gamma}{1 + \operatorname{ch}^2 \gamma}$.

2.

a) $2 \cdot b^2 \cdot x + 2 \cdot a^2 \cdot y \cdot y^{'} = 0 \Rightarrow y^{'} = -\frac{b^2 \cdot x}{a^2 \cdot y}$;

b) $\frac{\sqrt{x}}{2 \cdot x} + \frac{\cos y \cdot y^{'}}{3 \cdot \sqrt[3]{\sin^2 y}} = 0 \Rightarrow y^{'} = -\frac{3 \cdot \sqrt[6]{x^3 \cdot \sin^4 y}}{2 \cdot x \cdot \cos y}$;

c) $2 \cdot y \cdot y^{'} = \frac{(1+y^{'}) \cdot (x-y) - (x+y) \cdot (1-y^{'})}{(x-y)^2} \Rightarrow 2 \cdot y \cdot y^{'} = \frac{2 \cdot x \cdot y^{'} - 2 \cdot y}{(x-y)^2} \Rightarrow y^{'} = \frac{y}{x - y \cdot (x-y)^2}$;

d) $-\frac{1}{\sin^2 x} = y^2 + 2 \cdot x \cdot y \cdot y^{'} \Rightarrow y^{'} = -\frac{y^2 \cdot \sin^2 x + 1}{2 \cdot x \cdot \sin^2 x \cdot y}$;

e) $\frac{1}{x} + e^{-\frac{y}{x}} \cdot \left(\frac{1}{x^2} \cdot y - \frac{1}{x} \cdot y^{'}\right) = 0 \Rightarrow y^{'} = \frac{x + e^{-\frac{y}{x}} \cdot y}{e^{-\frac{y}{x}} \cdot x} = e^{\frac{y}{x}} + \frac{y}{x}$;

f) $\frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{1}{x} \cdot y^{'} - \frac{1}{x^2} \cdot y\right) = \frac{1}{2} \cdot \frac{2 \cdot x + 2 \cdot y \cdot y^{'}}{x^2 + y^2} \Rightarrow y^{'} = \frac{x+y}{x-y}$.

3.

a) $\ln\left[f_1(x)\right]=2\cdot\ln(x-2)-4\cdot\ln(x-1)-3\cdot\ln(x-3)\Rightarrow\dfrac{f_1^{'}(x)}{f_1(x)}=\dfrac{2}{x-2}-\dfrac{4}{x-1}-\dfrac{3}{x-3}\Rightarrow f_1^{'}(x)=\dfrac{(2-x)\cdot(5\cdot x^2-21\cdot x+24)}{(x-1)^5\cdot(x-3)^4};$

b) $\ln\left[f_2(t)\right]=\ln t+\dfrac{3}{4}\cdot\ln t-\dfrac{1}{4}\cdot\ln(t^2+1)\Rightarrow\dfrac{f_2^{'}(t)}{f_2(t)}=\dfrac{1}{t}+\dfrac{3}{4\cdot t}-\dfrac{t}{2\cdot(t^2+1)}\Rightarrow f_2^{'}(t)=\dfrac{(5\cdot t^2+7)\cdot\sqrt[4]{(t^3+t)^3}}{4\cdot(t^2+1)^2};$

c) $\ln\left[f_3(u)\right]=u\cdot\ln u\Rightarrow\dfrac{f_3^{'}(u)}{f_3(u)}=\ln u+u\cdot\dfrac{1}{u}\Rightarrow f_3^{'}(u)=u^u\cdot(1+\ln u);$

d) $\ln\left[f_4(v)\right]=\sqrt[3]{v^2}\cdot\ln v\Rightarrow\dfrac{f_4^{'}(v)}{f_4(v)}=\dfrac{2\cdot\ln v}{3\cdot\sqrt[3]{v}}+\dfrac{1}{\sqrt[3]{v}}\Rightarrow f_4^{'}(v)=\dfrac{1}{3}\cdot v^{\sqrt[3]{v^2}-\frac{1}{3}}\cdot(2\cdot\ln v+3);$

e) $\ln\left[f_5(w)\right]=(\sin w)\cdot\ln(\cos w)\Rightarrow\dfrac{f_5^{'}(w)}{f_5(w)}=(\cos w)\cdot\ln(\cos w)-(\sin w)\cdot\dfrac{\sin w}{\cos w}\Rightarrow$

$f\,'(x)=(\cos w)^{\sin w}\cdot\left[(\cos w)\cdot\ln(\cos w)-(\sin w)\cdot\operatorname{tg}w\right];$

f) $\ln\left[f_6(y)\right]=y\cdot\ln(y+2)-y\cdot\ln y\Rightarrow\dfrac{f_6^{'}(y)}{f_6(y)}=\ln(y+2)+\dfrac{y}{y+2}-\ln y-y\cdot\dfrac{2}{y}\Rightarrow f_6^{'}(y)=\left(1+\dfrac{2}{y}\right)^y\cdot\left[\ln\left(1+\dfrac{2}{y}\right)-\dfrac{2}{y+2}\right].$

4.

a) $(\cos x)^{'}=\left(\sin\left(\dfrac{\pi}{2}-x\right)\right)^{'}=\cos\left(\dfrac{\pi}{2}-x\right)\cdot\underbrace{\left(\dfrac{\pi}{2}-x\right)^{'}}_{=-1}=-\sin x;$

b) $f(x)=\sin x,\ f^{-1}(x)=\arcsin x\Rightarrow(\arcsin x)^{'}=\dfrac{1}{\cos(\arcsin x)}=\dfrac{1}{\sqrt{1-\sin^2(\arcsin x)}}=\dfrac{1}{\sqrt{1-x^2}};$

c) $f(y)=\cos y,\ f^{-1}(y)=\arccos y\Rightarrow(\arccos y)^{'}=-\dfrac{1}{\sin(\arccos y)}=-\dfrac{1}{\sqrt{1-\cos^2(\arccos y)}}=-\dfrac{1}{\sqrt{1-y^2}};$

d) $f(t)=\operatorname{tg}t,\ f^{-1}(t)=\operatorname{arctg}t\Rightarrow(\operatorname{arctg}t)^{'}=\dfrac{1}{\cos^2(\operatorname{arctg}t)}=\dfrac{1}{1+\operatorname{tg}^2(\operatorname{arctg}t)}=\dfrac{1}{1+t^2};$

e) $f(u)=\operatorname{ctg}u,\ f^{-1}(u)=\operatorname{arcctg}u\Rightarrow(\operatorname{arcctg}u)^{'}=-\dfrac{1}{\sin^2(\operatorname{arcctg}u)}=-\dfrac{1}{1+\operatorname{ctg}^2(\operatorname{arcctg}u)}=-\dfrac{1}{1+u^2};$

f) $f(v)=e^v,\ f^{-1}(v)=\ln v\Rightarrow(\ln v)^{'}=\dfrac{1}{e^{\ln v}}=\dfrac{1}{v}.$

5.

a) $y^{'}=\dfrac{2\cdot t}{4}=\dfrac{1}{2}\cdot t;$

b) $y^{'}=\dfrac{-\dfrac{4\cdot t}{(t^2+1)^2}}{-\dfrac{2\cdot(t^2-1)}{(t^2+1)^2}}=\dfrac{2\cdot t}{t^2-1};$

c) $y^{'}=\dfrac{\dfrac{2}{3\cdot\sqrt[3]{t}}}{\dfrac{1}{2\cdot\sqrt{t}}}=\dfrac{4}{3}\cdot\sqrt[6]{t};$

d) $y^{'}=\dfrac{3\cdot b\cdot\sin^2 t\cdot\cos t}{(-3)\cdot a\cdot\cos^2 t\cdot\sin t}=-\dfrac{b}{a}\cdot\operatorname{tg}t;$

e) $y^{'}=\dfrac{2\cdot e^{2\cdot t}}{-e^{-t}}=-2\cdot e^{3\cdot t};$

f) $y^{'}=\dfrac{e^t\cdot\sin t+e^t\cdot\cos t}{e^t\cdot\cos t-e^t\cdot\sin t}=\dfrac{\cos t+\sin t}{\cos t-\sin t}.$