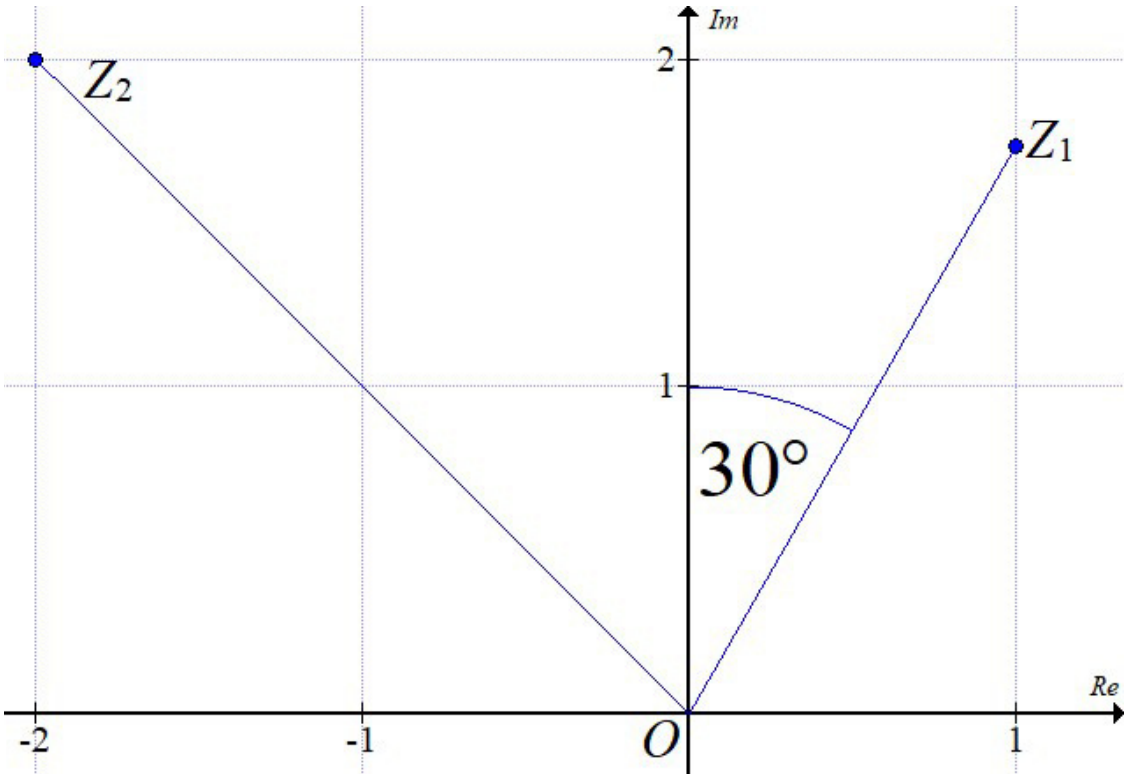
TVZ TEHNIČKO VELEUČILIŠTE U ZAGREBU POLYTECHNICUM ZAGRABIENSE Elektrotehnički odjel	**Matematika 1** (preddiplomski stručni studij elektrotehnike)	**Zadaci za grupne konzultacije** nastavne grupe **A i B** **5. 11. 2022.**

1. Kompleksnim brojevima z_1 i z_2 pridružene su redom točke Z_1 i Z_2 na slici 1. Odredite **glavni argument** kompleksnoga broja $z_3 = \left(\frac{z_1}{z_2}\right)^6$.



Slika 1.

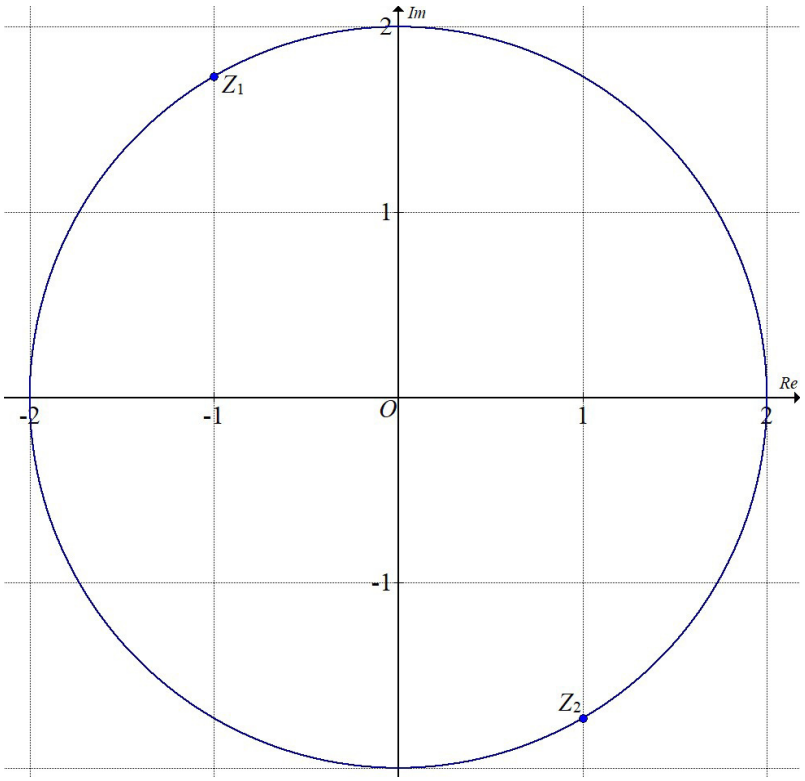
Rješenje: Iz zadane slike vidimo da su

$$\operatorname{Arg}(z_1) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3},$$
$$\operatorname{Arg}(z_2) = \pi - \operatorname{arctg}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3}{4}\cdot\pi.$$

Zbog toga je:

$$\operatorname{Arg}\left(\left(\frac{z_1}{z_2}\right)^6\right) = 6\cdot(\operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)) =$$
$$= 6\cdot\left(\frac{\pi}{3} - \frac{3}{4}\cdot\pi\right) =$$
$$= 6\cdot\left(\frac{-5}{12}\cdot\pi\right) = \frac{-5}{2}\cdot\pi = \frac{3}{2}\cdot\pi \text{ rad}.$$

2. Točkama Z_1 i Z_2 u Gaussovoj ravnini sa slike 2. pridruženi su redom brojevi z_1 i z_2. Odredite **glavni argument** kompleksnoga broja $z_3 = \left(\frac{1}{2}\cdot z_1\right)^{2021} + \overline{z_2}$.



Slika 2.

Rješenje: Zapišimo najprije broj z_1 u trigonometrijskom obliku. Imamo:

$$r_1 = 2,\ \mathrm{Re}(z_1) = -1 \Rightarrow$$
$$\mathrm{Arg}(z_1) = \pi - \arccos\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2}{3}\cdot\pi \Rightarrow$$
$$z_1 = 2\cdot\mathrm{cis}\left(\frac{2}{3}\cdot\pi\right).$$

Nadalje, očitamo:

$$r_2 = 2,\ \mathrm{Re}(z_2) = 1 \Rightarrow$$
$$\mathrm{Im}(z_2) = -\sqrt{2^2 - 1^2} = -\sqrt{4-1} = -\sqrt{3} \Rightarrow$$
$$z_2 = 1 - \sqrt{3}\cdot i \Leftrightarrow$$
$$\overline{z_2} = 1 + \sqrt{3}\cdot i.$$

Primjenom de Moivréove formule za potenciranje kompleksnoga broja dobivamo:

$$z_3 = \left(\frac{1}{2}\cdot z_1\right)^{2021} + \overline{z_2} =$$

$$= \left(\frac{1}{2}\cdot 2\cdot \operatorname{cis}\left(\frac{2}{3}\cdot\pi\right)\right)^{2021} + \left(1+\sqrt{3}\cdot i\right) =$$

$$= \left(\operatorname{cis}\left(\frac{2}{3}\cdot\pi\right)\right)^{2021} + 1+\sqrt{3}\cdot i =$$

$$= \operatorname{cis}\left(\frac{2}{3}\cdot\pi\cdot 2021\right) + 1+\sqrt{3}\cdot i =$$

$$= \cos\left(\frac{4042}{3}\cdot\pi\right) + i\cdot\sin\left(\frac{4042}{3}\cdot\pi\right) + 1+\sqrt{3}\cdot i =$$

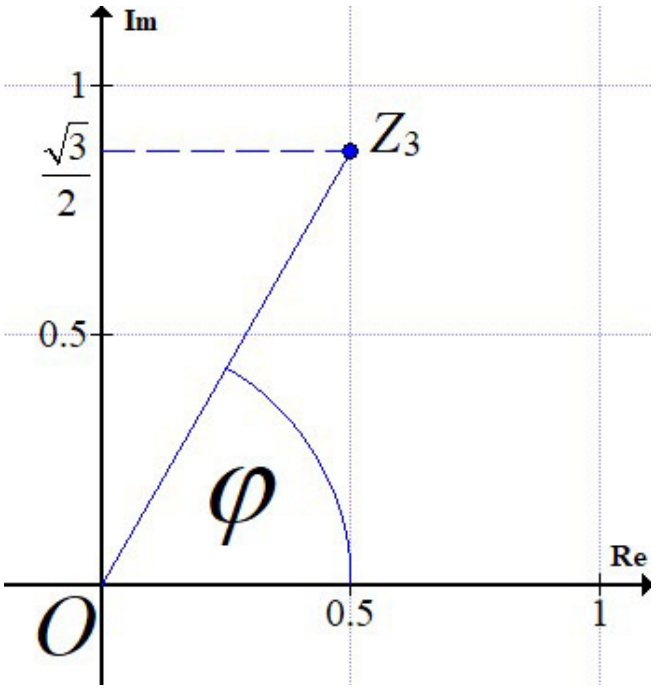
$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot i + 1 + \sqrt{3}\cdot i =$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot i.$$

Zbog toga je (vidjeti sliku 3.):

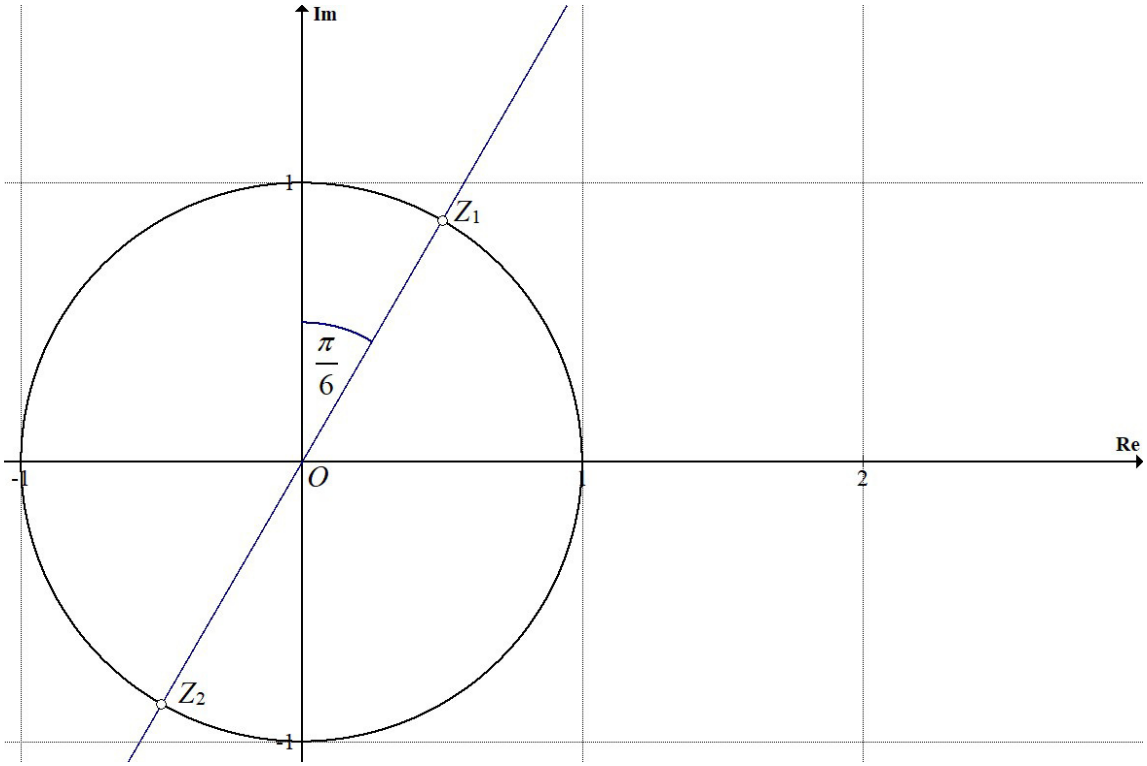
$$Z_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow$$

$$\operatorname{Arg}\left(z_3\right) = \operatorname{arctg}\left(\sqrt{3}\right) = \frac{\pi}{3} \text{ rad.}$$



Slika 3.

3. Točkama Z_1 i Z_2 u Gaussovoj ravnini sa slike 4. pridruženi su redom brojevi z_1 i z_2. Odredite $\operatorname{Arg}\left(z_1^{2019}+\overline{z_2}\right)$.



Slika 4.

Rješenje: Zapišimo najprije oba zadana broja u trigonometrijskom obliku. Imamo:

$$r_1 = 1,\ \operatorname{Arg}(z_1) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \Rightarrow$$

$$z_1 = \operatorname{cis}\left(\frac{\pi}{3}\right),$$

$$r_2 = 1,\ \operatorname{Arg}(z_2) = \pi + \frac{\pi}{3} = \frac{4}{3}\cdot\pi \Rightarrow$$

$$z_2 = \operatorname{cis}\left(\frac{4}{3}\cdot\pi\right).$$

Primjenom de Moivréove formule za potenciranje kompleksnoga broja dobivamo:

$$z_1^{2019}+\overline{z_2}=\operatorname{cis}\left(\frac{\pi}{3}\cdot 2019\right)+\operatorname{cis}\left(2\cdot\pi-\frac{4}{3}\cdot\pi\right)=$$

$$=\operatorname{cis}(673\cdot\pi)+\operatorname{cis}\left(\frac{2}{3}\cdot\pi\right)=$$

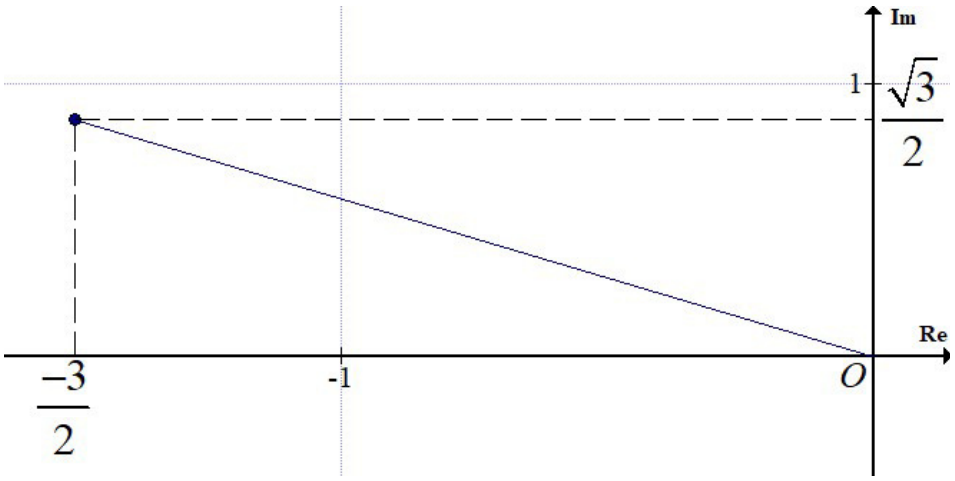
$$=\cos(673\cdot\pi)+i\cdot\sin(673\cdot\pi)+\cos\left(\frac{2}{3}\cdot\pi\right)+i\cdot\sin\left(\frac{2}{3}\cdot\pi\right)=$$

$$=-1+0\cdot i+\left(-\frac{1}{2}\right)+\frac{\sqrt{3}}{2}\cdot i=$$

$$=\frac{-3}{2}+\frac{\sqrt{3}}{2}\cdot i\Rightarrow$$

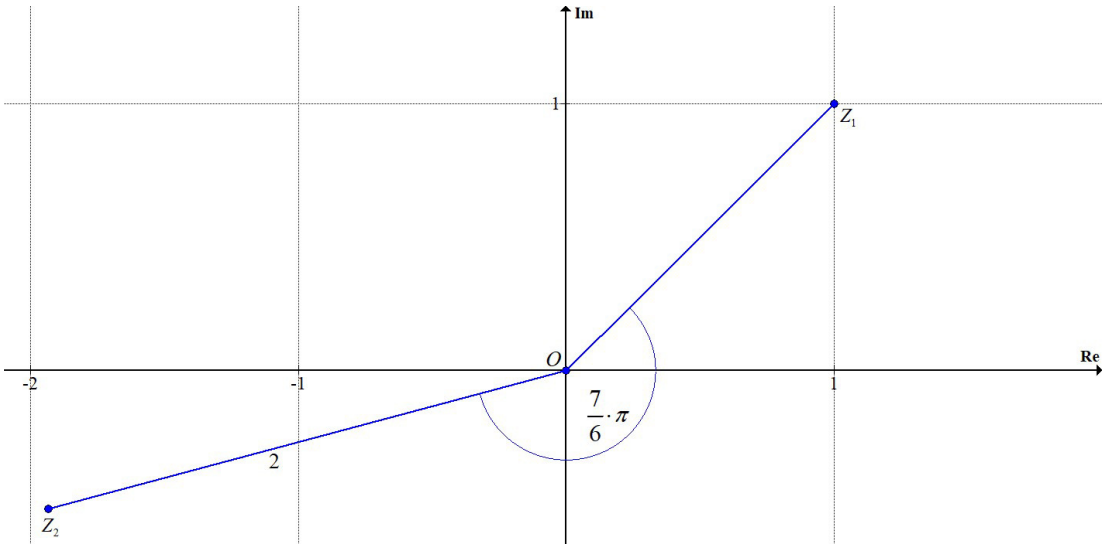
pripadna točka: $\left(\frac{-3}{2},\frac{\sqrt{3}}{2}\right)$ nalazi se u II. kvadrantu $\Rightarrow$

$$\operatorname{Arg}\left(z_1^{2019}+\overline{z_2}\right)=\pi-\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)=\pi-\frac{\pi}{6}=\frac{5}{6}\cdot\pi.$$



Slika 5.

4. Točkama Z_1 i Z_2 u Gaussovoj ravnini sa slike 6. pridruženi su redom brojevi z_1 i z_2. Izračunajte $\dfrac{z_1^{2030}}{z_2^{1014}}$ i zapišite dobiveni rezultat u algebarskom obliku.



Slika 6.

Rješenje: Zapišimo najprije oba broja u eksponencijalnom obliku. Imamo:

$$Z_1 = (1,\ 1) \Rightarrow$$

$$r_1 = \sqrt{1^2+1^2} = \sqrt{2},\quad \varphi_1 = \operatorname{arctg}\left(\frac{1}{1}\right) = \frac{\pi}{4} \Rightarrow$$

$$z_1 = \sqrt{2}\cdot \mathrm{e}^{i\cdot\frac{\pi}{4}};$$

$$r_2 = 2,\quad \varphi_2 = 2\cdot\pi - \left(\frac{7}{6}\cdot\pi - \frac{\pi}{4}\right) = \frac{13}{12}\cdot\pi \Rightarrow$$

$$z_2 = 2\cdot \mathrm{e}^{i\cdot\frac{13}{12}\cdot\pi}.$$

Primjenom de Moivréove formule za potenciranje kompleksnoga broja dobivamo:

$$\frac{z_1^{2030}}{z_2^{1014}} = \frac{\left(\sqrt{2}\right)^{2030}\cdot \mathrm{e}^{i\cdot\frac{\pi}{4}\cdot 2030}}{2^{1014}\cdot \mathrm{e}^{i\cdot\frac{13}{12}\cdot\pi\cdot 1014}} =$$

$$= \frac{\left(2^{\frac{1}{2}}\right)^{2030}}{2^{1014}}\cdot \mathrm{e}^{i\cdot\left(\frac{2030}{4}-\frac{13}{12}\cdot 1014\right)\cdot\pi} =$$

$$= \frac{2^{1015}}{2^{1014}}\cdot \mathrm{e}^{i\cdot(-591)\cdot\pi} =$$

$$= 2\cdot(\underbrace{\cos(-591\cdot\pi)}_{=-1} + i\cdot\underbrace{\sin(-591\cdot\pi)}_{=0}) = -2.$$

5. Zadan je kompleksan broj $z=\dfrac{1+\sqrt{3}\cdot i}{1-\sqrt{3}\cdot i}$. Odredite točku Gaussove ravnine pridruženu broju $\overline{z^{2019}}$.

Rješenje: Izračunajmo najprije broj z i zapišimo ga u trigonometrijskom obliku. Imamo:

$$z=\frac{1+\sqrt{3}\cdot i}{1-\sqrt{3}\cdot i}=\frac{\left(1+\sqrt{3}\cdot i\right)\cdot\left(1+\sqrt{3}\cdot i\right)}{\left(1-\sqrt{3}\cdot i\right)\cdot\left(1+\sqrt{3}\cdot i\right)}=$$

$$=\frac{1^2+2\cdot\sqrt{3}\cdot i+\left(\sqrt{3}\cdot i\right)^2}{1^2+\left(\sqrt{3}\right)^2}=$$

$$=\frac{1+2\cdot\sqrt{3}\cdot i-3}{3+1}=$$

$$=\frac{-2+2\cdot\sqrt{3}\cdot i}{4}=$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i,$$

$$r=|z|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1,$$

$$\varphi=\pi-\operatorname{arctg}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\pi-\operatorname{arctg}\left(\sqrt{3}\right)=\pi-\frac{\pi}{3}=\frac{2}{3}\cdot\pi,$$

$$z=r\cdot\operatorname{cis}\,\varphi=\operatorname{cis}\left(\frac{2}{3}\cdot\pi\right)$$

Primjenom de Moivréove formule za potenciranje kompleksnoga broja dobivamo:

$$z^{2019}=\left(\operatorname{cis}\left(\frac{2}{3}\cdot\pi\right)\right)^{2019}=\operatorname{cis}\left(\frac{2}{3}\cdot\pi\cdot 2019\right)=$$
$$=\operatorname{cis}\left(1346\cdot\pi\right)=\cos(1346\cdot\pi)+i\cdot\sin(1346\cdot\pi)=$$
$$=1+i\cdot 0=1.$$

Zbog toga je

$$\overline{z^{2019}}=\overline{1+0\cdot i}=1+0\cdot i=1.$$

Tom je broju pridružena točka (1, 0) i ta je točka rješenje zadatka.

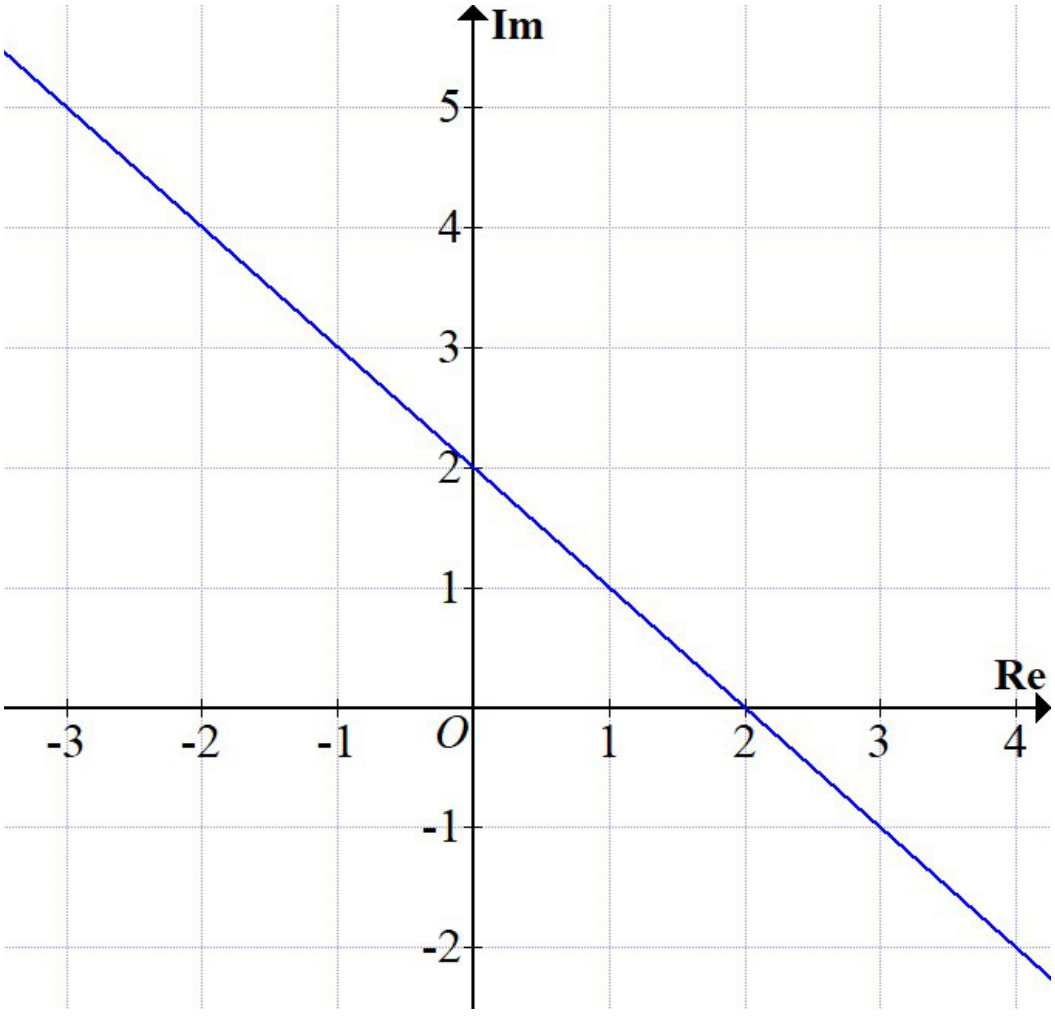
6. U Gaussovoj ravnini prikažite skup

$$S=\left\{z\in\mathbb{C}:2\cdot\mathrm{Re}(z)-\mathrm{Im}\left(2\cdot\overline{z}\right)=4.\right\}$$

Rješenje: Pretpostavimo da je $z=x+y\cdot i,\ x,y\in\mathbb{R}$. Tada imamo:

$$\begin{aligned}
&2\cdot\mathrm{Re}(z)-\mathrm{Im}\left(2\cdot\overline{z}\right)=4,\\
&2\cdot x-\mathrm{Im}\left(2\cdot(x-y\cdot i)\right)=4,\\
&2\cdot x-(-2\cdot y)=4,\\
&2\cdot x+2\cdot y=4,\\
&2\cdot y=-2\cdot x+4,\quad /:2\\
&y=-x+2.
\end{aligned}$$

Dakle, traženi prikaz je pravac čija je jednadžba $y=-x+2$ (vidjeti sliku 7.).



Slika 7.