The use of the computer program *Graph* in teaching application of differential calculus

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Abstract

The paper presents the authors' own teaching experience in the application of the computer program *Graph* in teaching unit *Problem of determining the tangent line and the normal line to the graph of real functions of one variable* within the mathematical courses in professional study programs of Zagreb University of Applied Sciences. The typical problems encountered during teaching the aforementioned unit and concrete examples how these problems can be successfully solved methodically are presented.

Keywords: *computer program Graph, differential calculus, tangent line, normal line, graphs of functions of one variable*

1. Introduction

One of the teaching units within the mathematical courses on the first year of professional studies of Zagreb University of Applied Sciences is *The problem of determining the tangent line and the normal line to the graph of real functions*. This teaching unit shows the typical applications of differential calculus of real functions of one variable. Within that unit, various tasks related to the tangent line and the normal line of the planar curve whose equation is given in explicit, implicit, or parametric form are solved.

The official curriculum of the mathematical courses bounds the teacher to show the students how to solve the mentioned tasks analytically, i.e. without the use of suitable computer

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programs. Possible sketches that are used in solving the tasks are related only to drawing tangent lines and normal lines as separated lines without any connection with the corresponding planar curve. However, experience in teaching, and especially the experience gained in the oral exams, shows that the analytical solving of these tasks must be accompanied with the corresponding graphical display of the planar curve, the tangent line and / or the normal line because most students have no idea what actually "happens" in that task or what is the relative position of the given curve to obtained tangent or normal line. Since in most tasks accurate sketching of the planar curve is technically relatively slow (requires determining domain, zeroes, monotony intervals etc.), methodically suitable computer program *Graph* is chosen to help.

2. The problem of (in)correct understanding of the concept of the tangent line to the planar curve

Based on the experience with teaching and oral examinations, we claim that the majority of students asked the question "What is a tangent line to a planar curve?" will answer "This is a line that intersects the given curve in exactly one point." Such a "definition" students met most likely during the secondary school education. By giving that answer, the students have in mind a situation like the one shown in Figure 1. In this figure the straight line intersects parabola in exactly one point (compare to [7]).

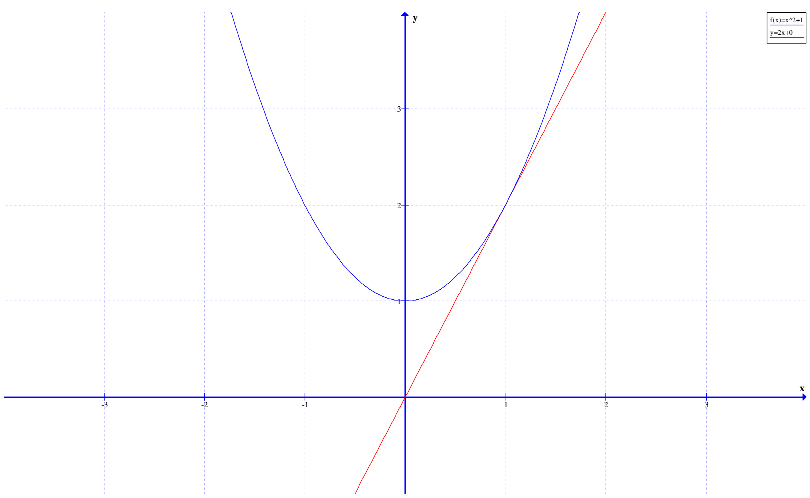

Figure 1. Parabola $y = x^2 + 1$ and a tangent line.

But when the students are asked whether the straight lines shown in Figures 2 and 3 are the tangent lined to the corresponding planar curves, their typical response is: "They are not, because they intersect the curves in more than one point." Even more, in the case shown in

Figure 3. they consider the question practically meaningless because, based on the figure, they conclude that the drawn line intersects the sinusoid in the infinite number of points.

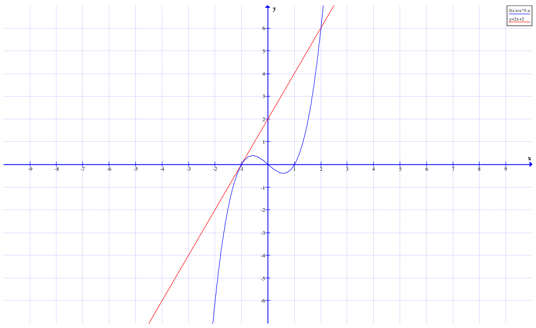

Figure 2. Curve $y = x^3 - x$ and its tangent line.

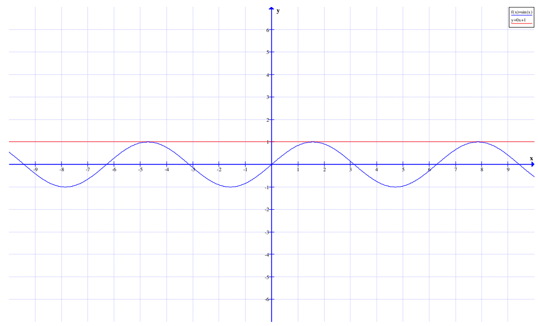

Figure 3. Sinusoid $y = \sin x$ and its tangent line.

Clearly, the above response is obviously wrong. In this article, we have no intention to discuss which definition of the concept of a tangent line is methodically the most appropriate, but we are using the following definition ([2], [5]):

***Definition* 1.** Let *f* be a real function of a real variable differentiable at a point x_0 and Γ_f the graph of the function *f*. Let us denote $y_0 := f(x_0)$. We say that the straight line *t* is the *tangent line* to Γ_f *at the point* $T = (x_0, y_0) \in \Gamma_f$ if the line *t* passes through the point *T* and the slope of the line equals $f'(x_0)$.

Note that the above definition says nothing about whether the line *t* intersects the graph of the function *f* in any other point (except point *T*), and that the definition does not apply to the planar curves which are not the graphs of a real function of one real variable (such as. circle, ellipse, etc.). However, taking into account the objective of teaching unit, and that is the application of differential calculus to solve the problem of determining the tangent and normal lines to the graph of real functions, we consider the above definition to be methodologically and pedagogically justified.

For completeness, we give the definition of a normal line given in accordance with the aim of the lesson stated above ([1], [3]):

***Definition* 2.** Let *f* be a real function of a real variable differentiable at a point x_0 and Γ_f the graph of the function *f*. Let us denote $y_0 := f(x_0)$. We say that the straight line *n* is the *normal line* to Γ_f *at the point* $T = (x_0, y_0) \in \Gamma_f$ if the line *n* passes through the point *T* and is perpendicular to the tangent line to Γ_f at the point *T*.

Remark 1. In the above definition we deliberately do not mention that the slope of the normal n equals to $k_n = -\frac{1}{f'(x_0)}$. This would require that the inequality $f'(x_0) \neq 0$ must be valid necessarily, which would complicate the definition of the normal line in any stationary point of f. Definitions above actually assume that the equation of *any* tangent line may be written in the form $y = kx + l$, where $k, l \in \mathbb{R}$, but that the equation of the normal line may be a form of $x = a$, for $a \in \mathbb{R}$.

The computer program *Graph* also applies the aforementioned assumptions. Before presenting its application, we give basic information about the program.

3. Basic information about the computer program *Graph*

A computer program *Graph* belongs to the open-source[2] computer programs intended primarily for drawing graphs of mathematical functions in a rectangular coordinate system in the plane. The functions may be given in the standard (explicit), parametric or polar form. Also, the program allows drawing of point series, tangent lines and normal lines to the curve at some point, cross-hatching to indicate the plane figures, etc. The interested reader may refer to [2]. The latest version of the program is available on the website of http://www.padowan.dk/download/ for free.

For the successful use of the program or setting function rules, it is enough to know the syntax that is used in MS Excel. Students learn working in MS Excel during their secondary education, so at the time of processing the lesson of the problem of determining tangents and normals they should have the necessary IT knowledge. We consider it appropriate to point out that the most of the students at the first year of professional studies at Zagreb University of Applied Sciences generally does not have (enough) experience with various programming languages, and by the time of processing the lesson of the problem of determining tangents and normals they still do not complete the course *Programming*. Therefore, the computer program in which this experience is not required is selected as methodically appropriate.

[2] Syntagma *open source* means the whole set of methods that are used in the computer applications development, with the entire development process and its results publicly available without special restrictions. This phrase is often (mistakenly!) translated to Croatian language as *application with open source*. This is only partly true, because the source code is actually the end result of the process of application development.

Working window of the program *Graph* is shown in Figure 4. How to use certain menus and options, we will show using concrete examples in the following section.

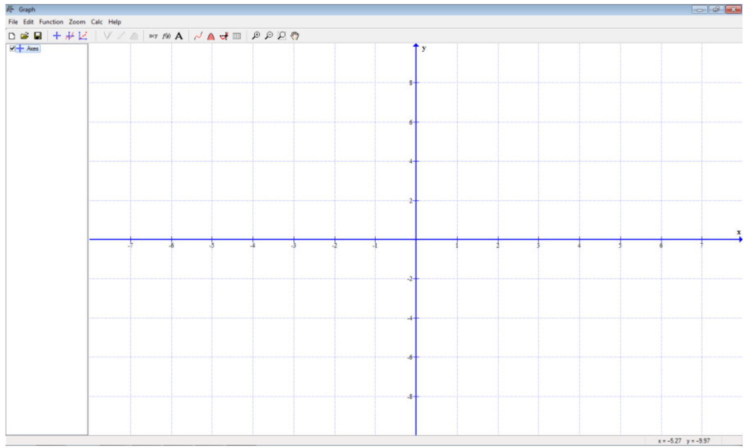

Figure 4. Working window of *Graph*.

4. Examples of the use of computer program *Graph*

To avoid increasing the volume of the text, we are giving only the end result of analytical solutions for each task. Details of task solving are left to the reader.

Example 1. Determine the equations of the tangent line and the normal line of the curve $y = x^2$ in the point $T = (1, y_T)$ of that curve ([1], [3], [6]).

Analytical solution: The equation of the tangent line: $t \ldots\ y = 2x - 1$.

The equation of the normal line: $n \ldots\ y = -\frac{1}{2}x + \frac{3}{2}$

Solution using the program Graph: Open the program *Graph*. Press the *Ins* key on the keyboard or choose the option *Insert function* from the drop-down menu *Function*. We get the dialog box shown in Figure 5.

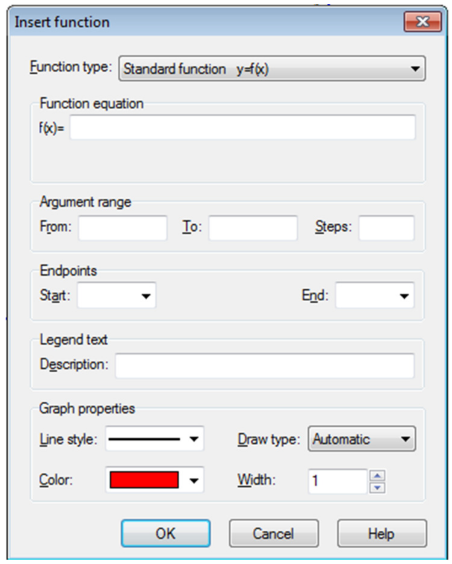

Figure 5. Dialog box for inserting a new function in *Graph*.

Click the mouse in the empty rectangle right next to the label f (x) =. We type: x ^ 2. All other rectangles are left unchanged. Then click OK. We get Figure 6.

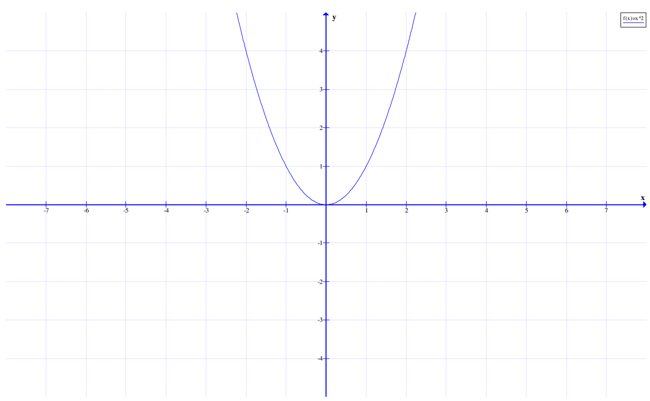

Figure 6. Curve $y = x^2$

Then press the F2 key or choose the option *Insert tangent/normal* from the drop-down menu *Function*. We get the dialog box shown in Figure 7.

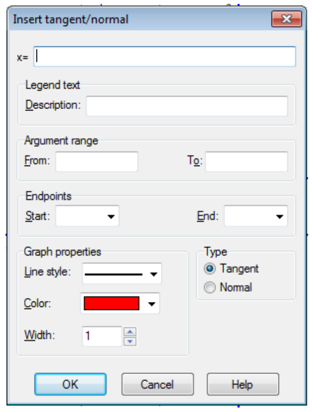

Figure 7. Dialog box for inserting tangent line or normal line in *Graph*.

Click the mouse in the rectangle right next to the label x =. The first coordinate of the point T equals 1, so we type 1 in this rectangle. First, we draw a tangent line and determine its equation. This option is already offered (as initial one) in the part of the box called *Type*, choose it and then click on OK. We get Figure 8.

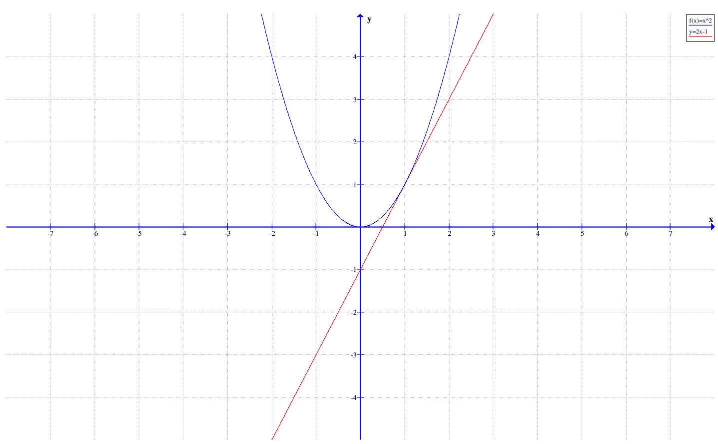

Figure 8. Curve $y = x^2$ and its tangent line at the point $T = (1, 1)$.

We can see the equation of the obtained tangent line in the legend in the upper right corner of the figure:

$$y = 2x - 1,$$

as obtained in the analytical solution. It remains to determine the equation of the normal line. For this purpose, we first click with the mouse on the equality f (x) = x ^ 2 shown on the left side of the working window (see Figure 9). By clicking on that equality we enable the determination of the equation of the normal line.

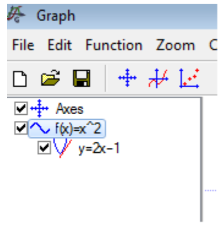

Figure 9. Choosing the curve equation before inserting its normal line.

Then we press the F2 key again. In the rectangle right next to the label x = we type 1, and in the part of the box below the label *Type* we click on a circle right next to the label *Normal* (See Figure 10). In order to distinguish the tangent line from the normal line, in part *Graph properties* of the graph we click on the dropdown menu next to the label *Color*, and choose eg. dark green color. Then we click OK.

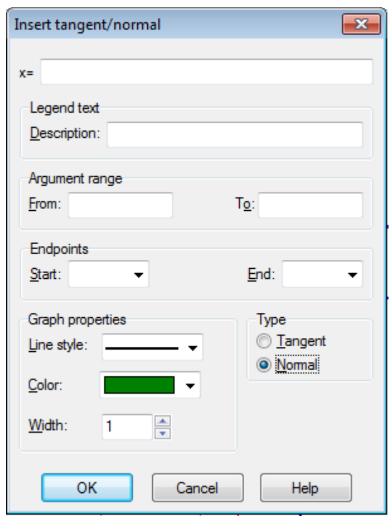

Figure 10. Entering the data needed for inserting the normal line and determining its equation.

We get Figure 11.

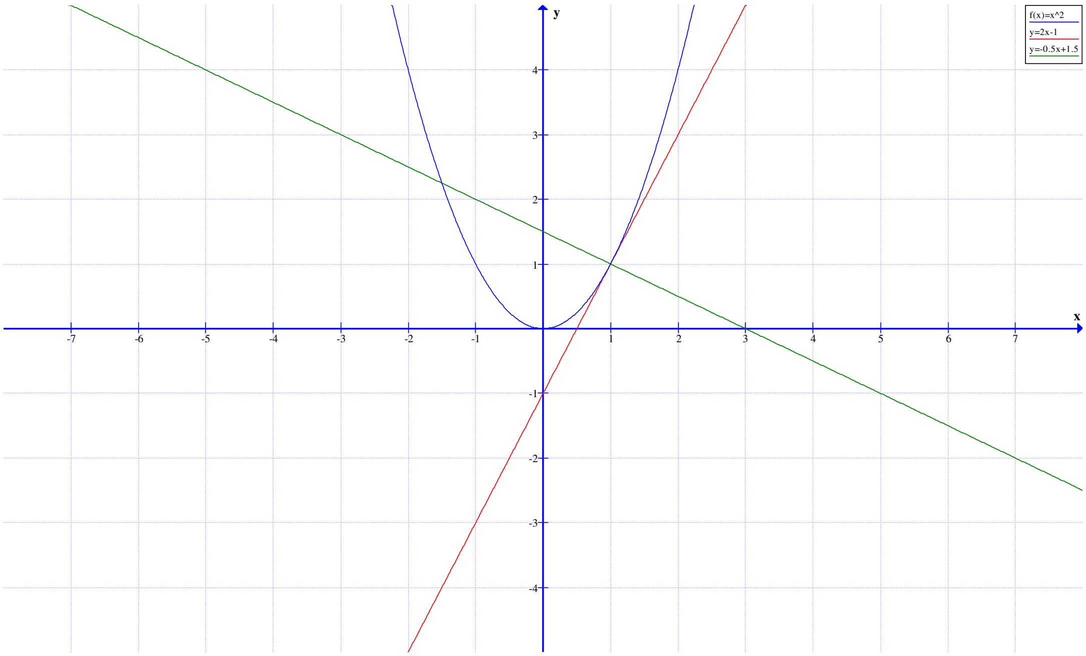

Figure 11. Solution of Example 1.

In the legend in the upper right corner we read the equation of the normal line:

$$y = -0.5x + 1.5.$$

It is easy to see that this equation is equivalent to the equation of the normal line obtained analytically. Note that the program *Graph* can not print fractions, so it replaces them with the appropriate decimal numbers.

Example 2. Determine the equations of the tangent line and the normal line of the curve $y=-x^2-4x-3$ in the point $T = (-2, y)$ of that curve ([1], [3]).

Analytical solution: The equation of the tangent line: $t \ldots\ y = 1$.

The equation of the normal line: $n \ldots\ x = -2$.

Solution using the program Graph:

We repeat the procedure analogous to that of the previous task. In the rectangle right next to the label f (x) we type: -x ^ 2-4x-3, and in the rectangle right next to the label x = in both cases we type -2. Finally we get Figure 12.

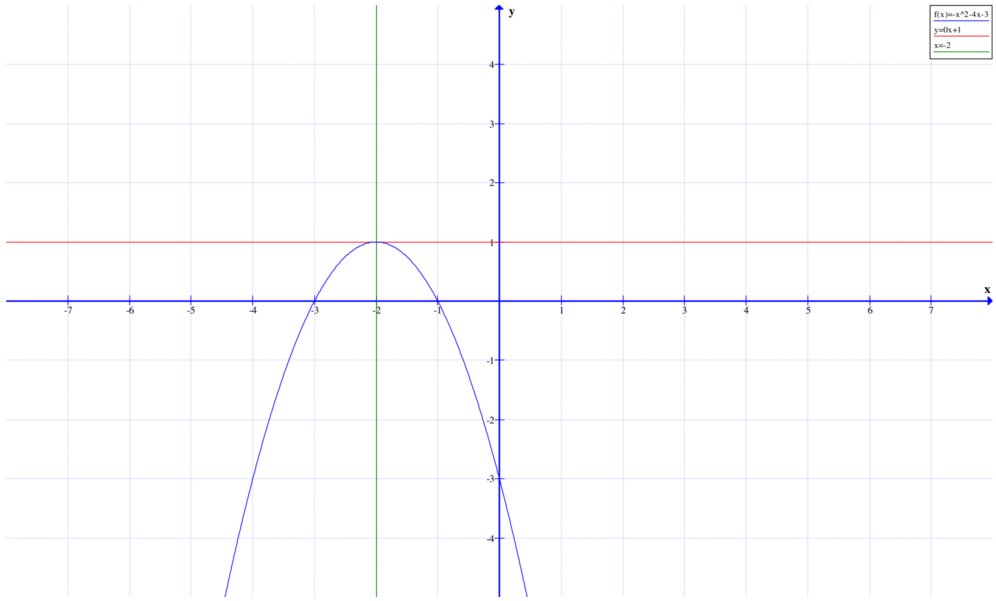

Figure 12. Solution of Example 2.

In the legend in the upper right corner we read the equations of the tangent line and the normal line:

$$y = 0x + 1,$$
$$x = -2.$$

We note that the equation of the tangent line is not completely simplified, i.e. it is not written in a simpler form $y = 1$. The reason for this is the previously mentioned assumption that the equation of the tangent line must necessarily be written in the form $y = kx + l$, where $k, l \in \mathbb{R}$.

Remark 2.

This task is important also because of the empirical experience of teachers, according to which many students enter the equation curve incorrect. The most common mistake is input `(-x)^2-4x-3` because students think that the equality $-x^2 = (-x)^2$ holds. Therefore, one of the purposes of this task methodically is emphasizing the validity of inequality $-x^2 \neq (-x)^2, \forall x \in \mathbb{R} \backslash \{0\}$.

Remark 3.

Unlike eg. MS Excel, *Graph* allows omission of character * for the multiplication of real numbers. That is why we can type the equation of the curve in Example 2. not using that character. Of course, the entry `-x^2-4*x-3` would also be correct.

Example 3. Determine the equations of the tangent line and the normal line of the curve $y=x^3-x$ in its intersection with the egative part of the axis abscissa ([1], [3]).

Analytical solution: The equation of the tangent line: $t... \ y = 2x + 2$.

The equation of the normal line: $n... \ y=-\frac{1}{2}x-\frac{1}{2}$.

Solution using Graph: We draw the curve using the procedure described in Example 1. We get Figure 13.

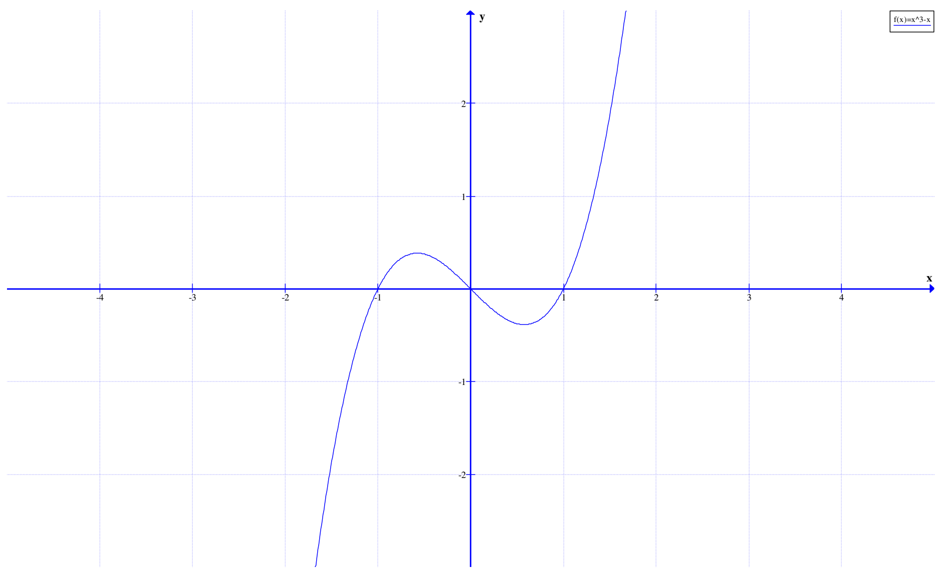

Figure 13. Curve $y = x^3 - x$

It is clear that the intersection of the curve with a negative part of the axis abscissa is the point $T=(-1,0)$. Therefore, we construct the tangent line and the normal line to the given curve at the point whose first coordinate is $x_0=-1$. Continuing the process of Example 1 we get Figure 14.

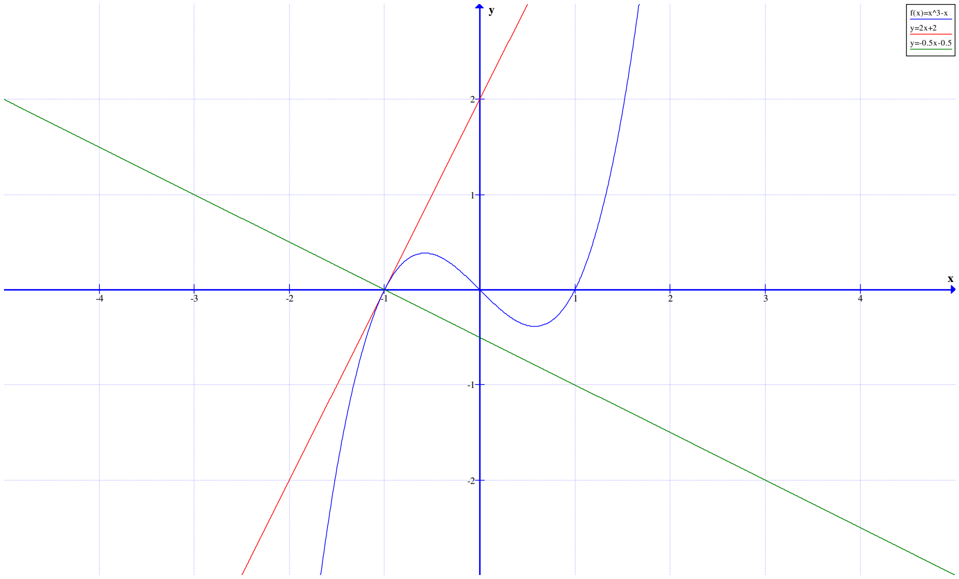

Figure 14. Solution of Example 3.

In the legend in the upper right corner we can see that the equations of the tangent line and the normal line are:

$$y = 2x + 2,$$
$$y = -0.5x - 0.5.$$

respectively.

At first glance, this example does not differ from Example 2. However, we use the possibility of the program *Graph* so that we get the graph of the curve over the segment [-2, 2]. For this purpose we use the icon  located on the toolbar. We click with the left mouse button a few times on that icon until the entire part of the curve over the segment [-2, 2] becomes visible. We get Figure 15.

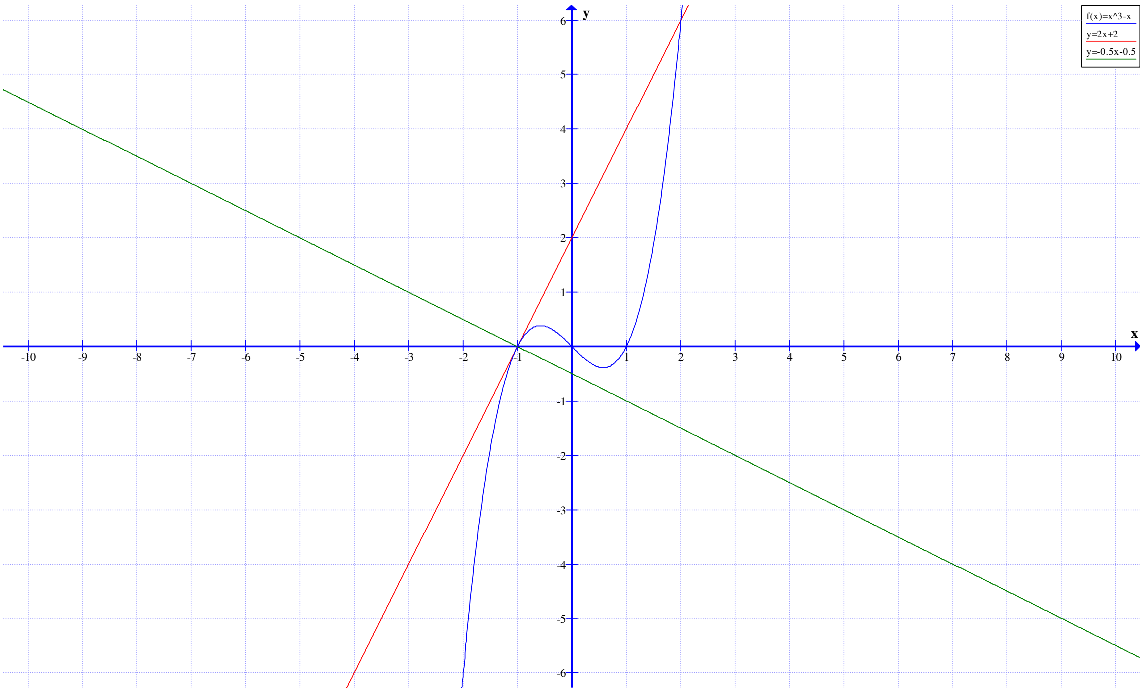

Figure 15. Curve $y = x^3 - x$ over the segment [–2, 2], tangent line and normal line to the curve at the point $T = (-1, 0)$.

The figure shows that the drawn tangent line intersects the curve also in the point $T_1 = (2, 6)$. According to the previous experience of students, obtained solution is incorrect because the line t cuts the given curve in exactly two points, and not in exactly one point. Therefore, the purpose of this example methodically is the illustration that the tangent line to the planar curve *can not be defined* as "the line which intersects the curve in exactly one point." Note that for the same purpose the example of a sinusoid and its tangent line to the point of local maximum is explained to students. That tangent line intersects sinusoid in infinitely many different points (see Figure 3).

In the last example, we show the determination of the equations of the tangent line and the normal line to planar curve given by parametric equations.

Example 4. The planar curve is given by parametric equations $\begin{cases} x = \sin t, \\ y = \cos t, \end{cases}$ for $t \in [0, 2\pi]$.

Determine the equations of the tangent line and the normal line to that curve at the point determined with the parameter $t = \dfrac{\pi}{2}$ ([1], [3]).

Analytical solution: The equation of the tangent line: $t \ldots\ x = 1$.

The equation of the normal line: $n \ldots\ y = 0$.

Solution using Graph: First, draw the given curve. Press the *Ins* key. In the drop-down menu next to the label *Function type*, select the option *Parametric function x (t), y (t)* (see Figure 16).

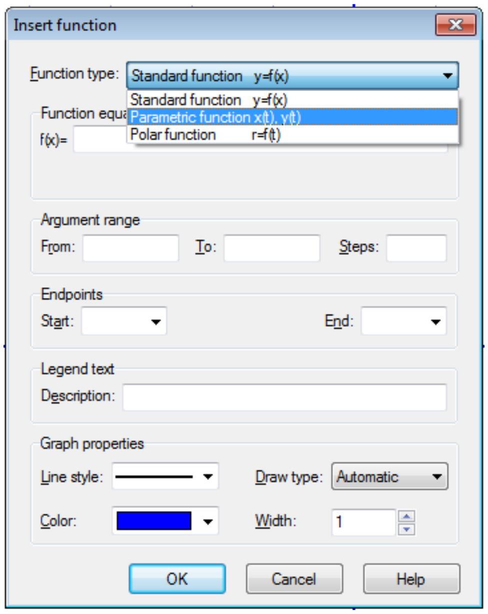

Figure 16. Choosing parametric function as function type.

We type `sin(t)` in the rectangle next to the label `x(t)=`, and `cos(t)` in the rectangle next to the label `y(t)=`. Then under the label *Argument range* we type 0 in the rectangle right next to the label *From:* and we type `2pi` in the rectangle right next to the label *To:* (see Figure 17). Then we click OK.

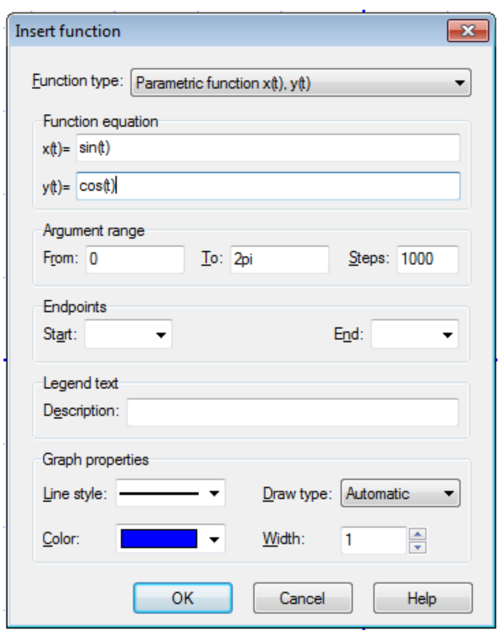

Figure 17. Entering the data for inserting the function from Example 4.

We determine the equations of the tangent line and the normal line using the procedure analogous to that described in Example 1, noting that in the menu *Insert tangent/normal* instead of x = it stands t =, so in the empty rectangle right next to that label should be written `pi/2`. All other steps of the mentioned procedures remain unchanged. So we get Figure 18.

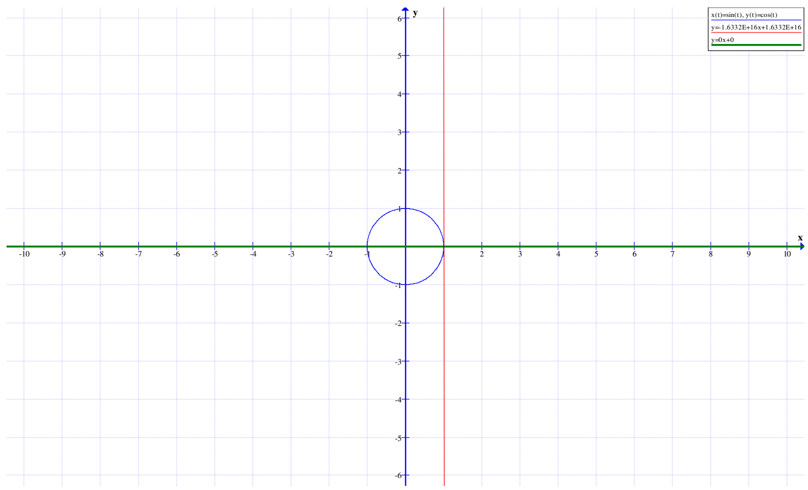

Figure 18. Solution of Example 4.

The solutions for the equations of the tangent line and the normal line are

$$y = -1.6332E + 16x + 1.6332E + 16,$$
$$y = 0x + 0,$$

respectively.

Note that the equation of the tangent line obtained using *Graph* does not coincide with the equation of the tangent line obtained analytically. This is because the slope of the tangent line "equals" $\dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$, and that numerical term is not defined. Even after the apparent cancellation of "values" -1.6332E and 1.6332 , the equation $y = 16x + 16$ is obtained which is obviously incorrect equation of the tangent line. Therefore, we can conclude that using *Graph* we can not determine the exact equation of the tangent line (to the curve given by parametric equations) which is parallel to the axis ordinate.

Remark 4. Example 4 can be solved correctly by selecting the polar form of the equation of the curve. The curve in the Example 4. is obviously central unit circle. Its equation in polar form is: $r = 1$. In the *Insert function* menu we choose the option *Polar function r = f (t)*, and we type

1 in the rectangle next to the label `r(t)=`. All other steps shown in Example 4. remain unchanged (including the entry of the values 0 and 2π). For parameter t we enter 0. So we get Figure 19.

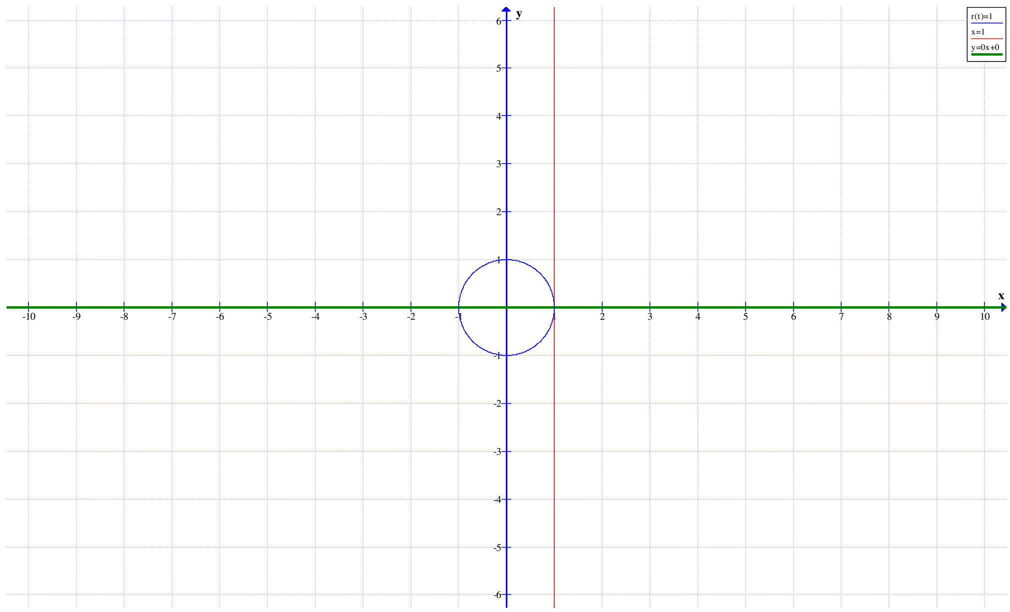

Figure 19. Solution of Example 4. in the case of polar form as function type.

We get the equations:

$$x = 1,$$
$$y = 0x + 0,$$

as obtained also in the analytical solution.

5. Conclusion

At the colleges and independent colleges in Croatia the use of different computer programs as aids in teaching mathematical courses is increasing. Due to very different levels of background IT-knowledge of students, those programs should be methodically suitable and enable the most of the users to learn how to use them quickly and easily. One of such programs is *Graph*. This paper describes the application of this program in teaching unit *Problem of determining the tangent line and the normal line to the graph of real function of one variable*. However, there are other possible applications of the same program on which this paper did not discuss (for example, calculate the length of a planar curve over a segment, calculate the surface area of some plane figures, etc.). We are confident that the application of such computer programs in teaching mathematical courses would contribute positively not only to improve the quality of teaching, but also to increase the interest of students to learn mathematics, and a better understanding of mathematical problems and ways of solving them. Solving certain kinds of problems using different methods, the comparison of solving methods and quality analysis of the obtained results are useful not only in mathematics, but also in many other areas of life.

Furthermore, apart from *Graph*, there are more free programs that teachers could use: *MathGV* (available at http://www.mathgv.com), *Desmos* (online program available at https://www.desmos.com/), and *Geogebra* (available at http://www.geogebra.org).

The program *MathGV* has a function syntax equal to the function syntax in *Graph*, allows to draw curves whose equations are given explicitly or parametrically, but does not allow the determination of a tangent line or a normal line equation in a point of a curve. These equations must first be determined analytically, and then draw a tangent line or a normal line as a separate function graph (without any link to the starting curve).

The program *Desmos*, as an online program, can be used exclusively on the Internet (it is not possible to download and install it on a PC, and use it without access to the Internet). Its syntax is somewhat simpler than the syntax of *Graph* (eg. the function argument does not need to be entered within brackets). It allows drawing tangent lines, but only giving the expression in the form $y = f^{'}(x_T) \cdot (x - x_T) + y_T$ where it is sufficient to define the function f and the point $T = (x_T, y_T)$, i.e. it is not necessary to enter the value for $f^{'}(x_T)$. Analogous statement is valid for drawing normal lines.

The program *Geogebra* allows the determination of the tangent line equation using the `Tangent` function. Unlike *Graph*, *Geogebra* also allows determining tangent lines parallel to the ordinate axis, and the determination of the tangent line equation of the curve given in the implicit form. However, the analogous statement is not valid for determining the normal line. The normal line may be determined using the function `PerpendicularLine` having the point T and the tangent line t as its arguments.

To conclude, we will cite [4]:

"When paper-pencil was used to learn about graphing, the teachers' role was a task setter and an explainer. On the other hand, when students used the software in their learning, teachers' role shifted to a consultant, a facilitator and a fellow investigator."

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Primjena računalnoga programa *Graph* u poučavanju primjene diferencijalnoga računa

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Sažetak

U radu se izlažu vlastita nastavna iskustva autora u primjeni računalnoga programa *Graph* u obradi nastavne cjeline *Problem određivanja tangente i normale na graf realne funkcije jedne realne varijable* u sastavu matematičkih predmeta na stručnim studijima Tehničkoga veleučilišta u Zagrebu. Navode se tipični problemi uočeni u obradi navedene nastavne cjeline, te na konkretnim primjerima objašnjava kako se ti problemi metodički mogu uspješno riješiti.

Ključne riječi: računalni program Graph, diferencijalni račun, tangenta, normala, grafovi funkcija jedne varijable.